ADVANCED PROBLEMS AND SOLUTIONS

Edited by

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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-430 *Proposed by Larry Taylor, Rego Park, NY*

Find integers j, $-2 < k < +2$, m_i and n_i such that:

(A) $5F_{m_i}F_{n_i} = L_k + L_{j+i}$, for $i = 1, 5, 9, 13, 17, 21$;

(B) $5F_{m_i}F_{n_i} = L_k - L_{j+i}$, for $i = 3, 7, 11, 15, 19, 23$;

(C) $F_{m_i}L_{n_i} = F_k + F_{j+i}$, for $i = 1, 2, \ldots, 22, 23$;

(D) $L_{m_i}F_{n_i} = F_k - F_{j+i}$, for $i = 1, 3, \ldots, 21, 23$;

(E) $L_{m_i}L_{n_i} = L_k - L_{j+i}$, for $i = 1, 5, 9, 13, 17, 21$;

(F) $L_{m_i}L_{n_i} = L_{-k} + L_{j+i}$, for $i = 2, 4, 6, 8$;

(G) $L_{m_i}L_{n_i} = L_k + L_{j+i}$, for $i = 3, 7, 11, 15, 16, 18, 19, 20, 22, 23$;

(H) $L_{m_i}L_{n_i} = L_k + F_{j+i}$, for $i = 10$;

(I) $L_{m_i}F_{n_i} = L_k + F_{j+i}$, for $i = 12$;

(J) $5F_{m_i}F_{n_i} = L_k + F_{j+i}$, for $i = 14$.

H-431 *Proposed by Piero Filipponi, Rome, Italy*

LA CATENA DI S. ANTONIO (St. Anthony's chain)

Let us consider a town having n (≥ 1) residents.

Step 1: One of them first draws out at random k ($1 \leq k \leq n$) distinct names from a directory containing the names of all town-dwellers (possibly, he/she may draw out also his/her own name), then he/she sends each of them an envelope containing one dollar.

Step 2: Every receiver (possibly, the sender himself/herself) acts as the sender.

Steps 3, 4, ...: As Step 2.

Find the probability $P_m(s, k, n)$ that, after s (≥ 1) steps, every town-dweller has received at least m (≥ 1) dollars.

Remark: It can readily be seen that

$$P_m(s,\ n,\ n) = 1 \text{ for } s = m,\ \lim_{s\to\infty} P_m(s,\ k,\ n) = 1, \text{ and}$$

$$P_1(s,\ k,\ n) = 0 \text{ if } s \le \begin{cases} n - 1 & (\text{for } k = 1) \\ \log_k(n(k - 1) + 1) - 1 & (\text{for } 1 < k < n). \end{cases}$$

H-432 *Proposed by Piero Filipponi, Rome, Italy*

For k and n nonnegative integers and m a positive integer, let $M(k,\ n,\ m)$ denote the arithmetic mean taken over the k^{th} powers of m consecutive Lucas numbers of which the smallest is L_n.

$$M(k,\ n,\ m) = \frac{1}{m}\sum_{j=n}^{n+m-1} L_j^k.$$

For $k = 2^h$ ($h = 0, 1, 2, 3$), find the smallest nontrivial value m_h ($m_h > 1$) of m for which $M(k,\ n,\ m)$ is integral for every n.

SOLUTIONS

Old Timer

H-365 *Proposed by Larry Taylor, Rego Park, NY*
(Vol. 22.1, February 1984)

Call a Fibonacci-Lucas identity divisible by 5 if every term of the identity is divisible by 5. Prove that, for every Fibonacci-Lucas identity not divisible by 5, there exists another Fibonacci-Lucas identity not divisible by 5 that can be derived from the original identity in the following way:

1) If necessary, restate the original identity in such a way that a derivation is possible.

2) Change one factor in every term of the original identity from F_n to L_n or from L_n to $5F_n$ in such a way that the result is also an identity. If the resulting identity is not divisible by 5, it is the derived identity.

3) If the resulting identity is divisible by 5, change one factor in every term of the original identity from L_n to F_n or from $5F_n$ to L_n in such a way that the result is also an identity. This is equivalent to dividing every term of the first resulting identity by 5. Then, the second resulting identity is the derived identity.

For example, $F_nL_n = F_{2n}$ can be restated as

$$F_nL_n = F_{2n} \pm F_0(-1)^n.$$

This is actually two distinct identities, of which the derived identities are

$$L_n^2 = L_{2n} + L_0(-1)^n$$

and

$$5F_n^2 = L_{2n} - L_0(-1)^n.$$

Partial solution by the proposer

1. Fibonacci-Lucas Equations

We define a Fibonacci-Lucas equation as an equation in one unknown in which one of the roots is equal to $(1 + \sqrt{5})/2$. Let $x = \sqrt{5}$ and $a = (1 + x)/2$. Let j

and k be integers, and let

$$A = F_j F_k, \quad B = F_j L_k, \quad C = L_j F_k, \quad D = L_j L_k, \quad E = F_{k+j},$$
$$F = F_{k-j}, \quad G = L_{k+j}, \quad H = L_{k-j}, \quad i = (-1)^j.$$

(Notice that F is not a Fibonacci number because it does not have a subscript.) Then, the following results are known:

$$5A = G - Hi,$$
$$B = E - Fi,$$
$$C = E + Fi,$$
$$D = G + Hi.$$

Let n be an integer. From

$$a^n = \frac{L_n + F_n x}{2}$$

and

$$xa^n = \frac{5F_n + L_n x}{2},$$

the following results can be obtained:

$$F_j a^k = (B + Ax)/2 \tag{1}$$
$$F_j xa^k = (5A + Bx)/2 \tag{2}$$
$$L_j a^k = (D + Cx)/2 \tag{3}$$
$$L_j xa^k = (5C + Dx)/2 \tag{4}$$
$$F_k a^j = (C + Ax)/2 \tag{5}$$
$$L_k a^j = (D + Bx)/2 \tag{6}$$
$$F_k xa^j = (5A + Cx)/2 \tag{7}$$
$$L_k xa^j = (5B + Dx)/2 \tag{8}$$

Subtracting (5) from (1), (6) from (2), (7) from (3), (8) from (4) gives:

$$(B - C)/2 = -Fi$$
$$(5A - D)/2 = -Hi$$
$$(D - 5A)/2 = Hi$$
$$(5C - 5B)/2 = 5Fi$$

After (4) minus (8) has been divided by x, this gives the following set of four Fibonacci-Lucas equations;

(E_1) $\quad F_j a^k - F_k a^j + F_{k-j}(-1)^j = 0$

(E_2) $\quad F_j xa^k - L_k a^j + L_{k-j}(-1)^j = 0$

(E_3) $\quad L_j a^k - F_k xa^j - L_{k-j}(-1)^j = 0$

(E_4) $\quad L_j a^k = L_k a^j - F_{k-j} x(-1)^j = 0$

2. Fibonacci-Lucas Identities

As yet, there is no rigorous definition of a Fibonacci-Lucas identity. Until such a definition is formulated, there will not be a complete solution of this problem. However, the following tentative definition can serve as the basis of a partial solution: Define a Fibonacci-Lucas identity as a rational form that can be derived from a Fibonacci-Lucas equation.

Let the letters A through L be redefined as follows:

$$A = F_j F_{n+k},\quad B = F_j L_{n+k},\quad C = L_j F_{n+k},\quad D = L_j L_{n+k},$$
$$E = F_k F_{n+j},\quad F = F_k L_{n+j},\quad G = L_k F_{n+j},\quad H = L_k L_{n+j},$$
$$I = F_{k-j} F_n,\quad J = F_{k-j} L_n,\quad K = L_{k-j} F_n,\quad L = L_{k-j} L_n.$$

After equations (E_1) through (E_4) have been multiplied by α^n, they can be restated as follows:

$$(B + Ax)/2 - (F + Ex)/2 + (J + Ix)/2i = 0$$
$$(5A + Bx)/2 - (H + Gx)/2 + (L + Kx)/2i = 0$$
$$(D + Cx)/2 - (5E + Fx)/2 - (L + Kx)/2i = 0$$
$$(D + Cx)/2 - (H + Gx)/2 - (5I + Jx)/2i = 0$$

Let

$$P_1 = A - E + Ii,\quad Q_1 = B - F + Ji,$$
$$P_2 = B - G + Ki,\quad Q_2 = 5A - H + Li,$$
$$P_3 = C - F - Ki,\quad Q_3 = D - 5E - Li,$$
$$P_4 = C - G - Ji,\quad Q_4 = D - H - 5Ii.$$

Then $(Q_t + P_t x)/2 = 0$ for $t = 1, 2, 3, 4$. But Q_t is a rational number. If $P_t \neq 0$, then $P x$ is an irrational number. The sum of a rational number and an irrational number cannot be equal to zero. Therefore, $P_t = 0$ and $Q_t = 0$.

It is clear that $P_t = 0$ and $Q_t = 0$ are rational forms that can be derived from a Fibonacci-Lucas equation. In the following set of eight Fibonacci-Lucas identities,

$P_1 = 0$ is equivalent to Identity (I_1);
$Q_1 = 0$ is equivalent to Identity (I_2);
$P_2 = 0$ is equivalent to Identity (I_3);
$Q_2 = 0$ is equivalent to Identity (I_4);
$P_3 = 0$ is equivalent to Identity (I_6);
$Q_3 = 0$ is equivalent to Identity (I_5);
$P_4 = 0$ is equivalent to Identity (I_7);
$Q_4 = 0$ is equivalent to Identity (I_8):

(I_1) $\quad F_j F_{n+k} = F_k F_{n+j} - F_{k-j} F_n (-1)^j$

(I_2) $\quad F_j L_{n+k} = F_k L_{n+j} - F_{k-j} L_n (-1)^j$

(I_3) $\quad F_j L_{n+k} = L_k F_{n+j} - L_{k-j} F_n (-1)^j$

(I_4) $\quad 5F_j F_{n+k} = L_k L_{n+j} - L_{k-j} L_n (-1)^j$

(I_5) $\quad L_j L_{n+k} = 5F_k F_{n+j} + L_{k-j} L_n (-1)^j$

(I_6) $\quad L_j F_{n+k} = F_k L_{n+j} + L_{k-j} F_n (-1)^j$

(I_7) $\quad L_j F_{n+k} = L_k F_{n+j} + F_{k-j} L_n (-1)^j$

(I_8) $\quad L_j L_{n+k} = L_k L_{n+j} + 5F_{k-j} F_n (-1)^j$

It can be observed that Identities (I_1) through (I_8) are not divisible by 5. Also note that (I_1) and (I_2) can be derived from each other by the method described in this problem, (I_3) and (I_4) can be derived from each other, (I_6) and (I_5) can be derived from each other, and (I_7) and (I_8) can be derived from each other.

A Prime Example

H-408 *Proposed by Robert Shafer, Berkeley, CA*
(Vol. 25.1, February 1987)

(a) Define $u_0 = 3$, $u_1 = 0$, $u_2 = 2$, and $u_{n+1} = u_{n-1} + u_{n-2}$ for all integers n.

(b) In addition, let $w_0 = 3$, $w_1 = 0$, $w_2 = -2$, and $w_{n+1} = -w_{n-1} + w_{n-2}$ for all integers n.

Prove:

$$u_p \equiv w_p \equiv 0 \pmod{p} \quad \text{and} \quad u_{-p} \equiv -w_{-p} \equiv -1 \pmod{p},$$

where p is a prime number.

Solution by C. Georghiou, Patras, Greece

We need the following lemma.

Lemma: Let

$$f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$$

be a monic polynomial with integer coefficients and denote its roots by r_1, r_2, ..., r_n. Then, for any prime p,

$$r_1^p + r_2^p + \cdots + r_n^p \equiv (r_1 + r_2 + \cdots + r_n)^p \pmod{p}.$$

Proof: First, it is easy to see that if p is prime, then the multinomial coefficient

$$\frac{p!}{k_1!k_2! \ldots k_n!}$$

is divisible by p when $0 \le k_i < p$, $i = 1, 2, \ldots, n$, and $k_1 + k_2 + \cdots + k_n = p$. Second, by the multinomial theorem, we have

$$(r_1 + r_2 + \cdots + r_n)^p = \sum_{k_1+k_2+\cdots+k_n=p} \binom{p}{k_1, k_2, \ldots, k_n} r_1^{k_1} r_2^{k_2} \ldots r_n^{k_n}$$

$$= r_1^p + r_2^p + \cdots + r_n^p + \sum_{\substack{0 \le k_1 \le k_2 \le \cdots \le k_n < p \\ k_1+k_2+\cdots+k_n=p}} \binom{p}{k_1, k_2, \ldots, k_n}$$

$$\times\, g_k(r_1, r_2, \ldots, r_n),$$

where

$$g_k \equiv g_{k_1 k_2 \ldots k_n}$$

is a symmetric polynomial with integer coefficients. Then, by the Fundamental Theorem on Symmetric FUnctions (see, e.g., C. R. Hadlock: *Field Theory and Its Classical Problems*, MAA Publ., 1978, p. 42), each $g_k(r_1, r_2, \ldots, r_n)$ can be written as a polynomial h_k in the elementary symmetric functions with integer coefficients. Since $g_k(r_1, r_2, \ldots, r_n)$ takes integer values, the lemma is established.

(a) From the initial conditions we find that, for $-\infty < n < +\infty$,

$$u_n = r_1^n + r_2^n + r_3^n$$

where r_1, r_2, r_3 are the roots of the (irreducible) polynomial

$$f(x) = x^3 - x - 1.$$

Therefore, for any prime p,

$$u_p = r_1^p + r_2^p + r_3^p \equiv (r_1 + r_2 + r_3)^p = 0 \pmod{p};$$

$$u_{-p} = \left(\frac{1}{r_1}\right)^p + \left(\frac{1}{r_2}\right)^p + \left(\frac{1}{r_3}\right)^p \equiv \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)^p = (-1)^p = -1 \pmod{p}.$$

(b) Same as above, with r_1, r_2, r_3 the roots of the (irreducible) polynomial

$$f(x) = x^3 + x - 1,$$

and we find that

$$w_p = r_1^p + r_2^p + r_3^p \equiv (r_1 + r_2 + r_3)^p = 0 \pmod{p};$$

$$w_{-p} = r_1^{-p} + r_2^{-p} + r_3^{-p} \equiv \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)^p = 1^p = 1 \pmod{p}.$$

Also solved by P. Bruckman and the proposer.

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