ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

A. P. Hillman

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-640 *Proposed by Russell Euler, Northwest Missouri State U., Marysville, MO*

Find the determinant of the $n \times n$ matrix (x_{ij}) with $x_{ij} = 1$ for $j = i$ and for $j = i - 1$, $x_{ij} = -1$ for $j = i + 1$, and $x_{ij} = 0$, otherwise.

B-641 *Proposed by Dario Castellanos, U. de Carabobo, Valencia, Venezuela*

Prove that

$$F_{mn} = \frac{1}{\sqrt{5}}\left[\left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n - \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n\right],$$

$$L_{mn} = \left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n + \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n.$$

B-642 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

It is known that

$$L_{2(2n+1)} = L_{2n+1}^2 + 2,$$

and it can readily be proven that

$$L_{3(2n+1)} = L_{2n+1}^3 + 3L_{2n+1}.$$

Generalize these identities by expressing $L_{k(2n+1)}$, for integers $k \ge 2$, as a polynomial in L_{2n+1}.

B-643 *Proposed by T. V. Padnakumar, Trivandrum, South India*

For positive integers a, n, and p, with p prime, prove that

$$\binom{n + ap}{p} - \binom{n}{p} \equiv a \pmod{p}.$$

B-644 *Proposed by H. W. Corley, U. of Texas at Arlington*

Consider three children playing catch as follows. They stand at the vertices of an equilateral triangle, each facing its center. When any child has the ball, it is thrown to the child on her or his left with probability 1/3 and to the child on the right with probability 2/3. Show that the probability that the initial holder has the ball after n tosses is

$$\frac{2}{3}\left(\frac{\sqrt{3}}{3}\right)^n \cos\left(\frac{5n\pi}{6}\right) + \frac{1}{3} \text{ for } n = 0, 1, 2, \ldots .$$

B-645 *Proposed by R. Tošić, U. of Novi Sad, Yugoslavia*

Let

$$G_{2m} = \binom{2m-1}{m} - 2\binom{2m-1}{m-3} + \binom{2m}{m-5} \text{ for } m = 1, 2, 3, \ldots,$$

$$G_{2m+1} = \binom{2m}{m} - \binom{2m+1}{m-2} + 2\binom{2m}{m-5} \quad \text{for } m = 0, 1, 2, \ldots,$$

where $\binom{n}{k} = 0$ for $k < 0$. Prove or disprove that $G_n = F_n$ for $n = 0, 1, 2, \ldots$.

SOLUTIONS

Cyclic Permutations Modulo 6 and Modulo 5

B-616 *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, MA*

(a) Find the smallest positive integer a such that

$$L_n \equiv F_{n+a} \pmod 6 \text{ for } n = 0, 1, \ldots .$$

(b) Find the smallest positive integer b such that

$$L_n \equiv F_{5n+b} \pmod 5 \text{ for } n = 0, 1, \ldots .$$

Solution by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

By inspection of the sequences $\{L_n\}$ and $\{F_n\}$ reduced modulo 6 (both with repetition period equal to 24), it is readily seen that $a = 6$.

By inspection of the above sequences reduced modulo 5 (repetition period equals 8 for $\{L_n\}$ and 20 for $\{F_n\}$), it is readily seen that $b = 3$.

Also solved by Paul S. Bruckman, Herta T. Freitag, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Fibonacci Parallelograms

B-617 *Proposed by Stanley Rabinowitz, Littleton, MA*

Let R be a rectangle each of whose vertices has Fibonacci numbers as its coordinates x and y. Prove that the sides of R must be parallel to the coordinate axes.

Solution taken from those by Paul S. Bruckman, Fair Oaks, CA and Philip L. Mana, Albuquerque, NM

It will be shown that the rectangle either has its sides parallel to the axes or it is a square whose sides have inclinations 45° and -45°.

Let (F_a, F_h), (F_b, F_i), (F_c, F_j), (F_d, F_k) be the vertices of a parellelogram in counterclockwise order. If its sides are not parallel to the axes, we may assume that

$$F_a < F_b < F_c \quad \text{and} \quad F_a < F_d < F_c. \tag{1}$$

Since the diagonals bisect each other,

$$F_a + F_c = F_b + F_d. \tag{2}$$

By (1), $c - a \geq 2$, so $F_a + F_c$ is a unique Zeckendorf representation. This, with (1) and (2), implies that $b = d$ and that $b = a + 2$ and $c = a + 3$.

Similarly, one has

$$F_i < F_h < F_j \quad \text{and} \quad F_i < F_j < F_k$$

and can show that $j = h = i + 2$ and $k = i + 3$. Now the slope of two sides is

$$\frac{F_i - F_h}{F_b - F_a} = \frac{F_i - F_{i+2}}{F_{a+2} - F_a} = -\frac{F_{i+1}}{F_{a+1}}$$

and the slope of the other sides is F_{i+1}/F_{a+1}. Thus, the parallelogram is a rectangle if and only if $F_{i+1}^2 = F_{a+1}^2$. This happens (for nonnegative subscripts) if and only if $F_{i+1} = F_{a+1}$. This, in turn, is true if and only if $i = a$ or $\{i, a\} = \{0, 1\}$. These cases give the rectangles with vertices

$$\begin{array}{llll} (F_a, F_{a+2}), & (F_{a+2}, F_a), & (F_{a+3}, F_{a+2}), & (F_{a+2}, F_{a+3}); \\ (0, 2), & (1, 1), & (2, 2), & (1, 3); \\ (2, 0), & (1, 1), & (2, 2), & (3, 1). \end{array}$$

Each of these is a square whose sides have inclinations 45° and -45°.

Counterexamples (that is, squares with sides not parallel to the axes) given by Piero Filipponi and Herta Freitag.

Multiples of 40

B-618 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S(n) = L_{2n+1} + L_{2n+3} + L_{2n+5} + \cdots + L_{4n-1}$. Prove that $S(n)$ is an integral multiple of 10 for all even positive integers.

Solution by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA

We prove a more general result, namely:

$$S(n) \equiv 0 \pmod{40} \text{ for all even positive integers } n.$$

Using Binet form for Lucas numbers with $L_m = \alpha^m + \beta^m$, we have:

$$\begin{aligned} S(n) &= \alpha^{2n+1}\sum_{i=0}^{n-1}\alpha^{2i} + \beta^{2n+1}\sum_{i=0}^{n-1}\beta^{2i} \\ &= \alpha^{4n} - \alpha^{2n} + \beta^{4n} - \beta^{2n} = L_{4n} - L_{2n}. \end{aligned}$$

Let $n = 2k$, then

$$S(2k) = L_{8k} - L_{4k} = 5F_{6k}F_{2k}, \text{ where } k \geq 1,$$

by using I_{16} and I_{25} in Hoggatt's *Fibonacci and Lucas Numbers*.

Since F_6 divides F_{6k}; we conclude that:

$$S(2k) \equiv 0 \pmod{40}.$$

Also solved by Paul S. Bruckman, David M. Burton, Piero Filipponi, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, and the proposer.

More Multiples of 10

B-619 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $T(n) = F_{2n+1} + F_{2n+3} + F_{2n+5} + \cdots + F_{4n-1}$. For which positive integers n is $T(n)$ an integral multiple of 10?

Solution by David M. Burton, U. of New Hampshire, Durham, NH

$T(n)$ is an integral multiple of 10 provided n is a multiple of 5. First, note that the identity

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$$

gives us $T(n) = F_{4n} - F_{2n}$.

Now

$$F_{4n} - F_{2n} \equiv 2n \text{ or } 4n \pmod 5,$$

according as n is odd or even; thus, $T(n) \equiv 0 \pmod{10}$ if and only if 5 divides n.

To see that $F_{4n} - F_{2n} \equiv 2n$ or $4n \pmod 5$, simply use the congruence

$$F_{2n} \equiv n(-1)^{n+1} \pmod 5.$$

[see the solution to Problem B-379 in the April 1979 issue], which yields

$$F_{4n} - F_{2n} \equiv 4n[2 - (-1)^n] \pmod 5.$$

This could equally well be derived from the congruence

$$F_{2n} = nL_n \pmod 5$$

[see the solution to Problem B-368 in the December 1978 issue], together with the two relations

$$L_{2n} = 5F_n^2 + 2(-1)^n \equiv 2 \text{ or } 3 \pmod 5,$$

$$L_{4n} = 5F_{2n}^2 + 2 \equiv 2 \pmod 5.$$

Also solved by Paul S. Bruckman, Piero Filipponi, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Congruence Modulo 9

B-620 *Proposed by Philip L. Mana, Albuquerque, NM*

Prove that $F_{24k+3}^n + F_{24k+5}^n \equiv 2F_{24k+6}^n \pmod 9$ for all n and k in $N = \{0, 1, 2, \ldots\}$.

Solution by Paul S. Bruckman, Fair Oaks, CA

The sequence $(F_n \pmod 9))_{n=0}^{\infty}$ is periodic with period 24, and the period is as follows:

$$(0,1,1,2,3,5,8,4,3,7,1,8,0,8,8,7,6,4,1,5,6,2,8,1).$$

Inspection of this period shows that:

$$F_{24k+3} \equiv 2,\ F_{24k+5} \equiv 5, \text{ and } F_{24k+6} \equiv 8 \pmod 9.$$

The problem is therefore equivalent to proving the congruence

$$2^n + 5^n \equiv 2 \cdot 8^n \pmod 9, \text{ for all } n. \tag{1}$$

We form the sequences

$$(2^n \pmod 9))_{n=0}^{\infty},\ (5^n \pmod 9))_{n=0}^{\infty}, \text{ and } (2 \cdot 8^n \pmod 9))_{n=0}^{\infty},$$

and find that these are all periodic of period 6; these periods are,

$$(1,2,4,8,7,5),\ (1,5,7,8,4,2), \text{ and } (2,7,2,7,2,7),$$

respectively (actually, the last sequence is periodic with only period 2, but we have triplicated the terms in order to make them compatible with those of the other two sequences). Therefore, we see that in all cases, the congruence in (1) is satisfied, proving the original problem.

Also solved by Odoardo Brugia & Piero Filipponi, Herta T. Freitag, L. Kuipers, Bob Prielipp, Sahib Singh, Lawrence Somer, and the proposer.

Powers of F_{2h} modulo F_{2h-1}

B-621 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $n = 2h - 1$ with h a positive integer. Also, let $K(n) = F_h L_{h-1}$. Find sufficient conditions on F_n to establish the congruence

$$F_{n+1}^{K(n)} \equiv 1 \pmod{F_n}.$$

Solution by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA

As $n + 1$ is even, therefore using I_{13} of Hoggatt's *Fibonacci and Lucas Numbers*, we have

$$F_n F_{n+2} = F_{n+1}^2 + 1 \Rightarrow F_{n+1}^2 \equiv -1 \pmod{F_n}.$$

Thus, the order of F_{n+1} modulo F_n is 4.

From the property of order, it follows that:

$$F_{n+1}^{F_h L_{h-1}} \equiv 1 \pmod{F_n} \text{ is true only when 4 divides } F_h L_{h-1}.$$

This is possible when 4 divides F_h or 4 divides L_{h-1}. (Since 2 is not a factor of F_h and also a factor of F_{h-1} for any h.)t possible for any h.)

$$4 \text{ divides } F_h \Rightarrow h = 6t \text{ or } n = 12t - 1.$$

$$4 \text{ divides } L_{h-1} \Rightarrow h - 1 = (2t - 1)3 \Rightarrow h = 6t - 2 \Rightarrow n = 12t - 5.$$

Thus, the required values of n are 1 and 3, together with those positive integers n which satisfy

$$n \equiv 7 \pmod{12} \quad \text{or} \quad n \equiv 11 \pmod{12}.$$

Also solved by Paul S. Bruckman, L. Kuipers, Bob Prielipp, Lawrence Somer, and the proposer.

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LETTER TO THE EDITOR

February 3, 1989

Dear Dr. Bergum,

I'd like to point out that some results which appeared in Michael Mays's recent article, "Iterating the Division Algorithm" [*Fib. Quart.* 25 (1987):204-213] were already known.

In particular, his Algorithm 6, which on input (b, a) sets $a = a$ and $a_{1} = b \bmod a$, appeared in my paper, "Metric Theory of Pierce Expansions," [*Fib. Quart.* 24 (1986):22-40]. His Theorem 4, proving that $L(b, a) \leq 2\sqrt{b} + 2$ [where $L(b, a)$ is the least n such that $a = 0$), appears in my paper as Theorem 19.

Let Ω, Ω' be defined as follows: we write $f(n) = \Omega(g(n))$ if there exist c, N such that $f(n) \geq cg(n)$ for all $n \geq N$. We write $f(n) = \Omega'(g(n))$ if there exists c such that $f(n) \geq cg(n)$ infinitely often. Since my paper appeared, I have proved

$$\max_{1 \leq a \leq n} L(n, a) = \Omega'(\log n)$$

and

$$\sum_{1 \leq a \leq n} L(n, a) = \Omega(n \log \log n).$$

The details are available to those interested.

Recently, I also stumbled across what may be the first reference to this type of algorithm. It is J. Binet, "Recherches sur la théorie des nombres entiers et sur la résolution de l'équation indéterminee du premier degré qui n'admet que des solutions entières," *J. Math. Pures Appl.* 6 (1841):449-494. Binet's algorithm, however, takes the *absolutely least residue* at each step, rather than the positive residue, and it is therefore easier to prove there are no long expansions.

Sincerely yours,

Jeffrey Shallit