OPERATIONS ON GENERATORS OF UNITARY AMICABLE PAIRS

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0. Introduction

McClung [3] defined a *generator* (of unitary amicable pairs) as a pair (f, k) where f is a rational not one and k and fk are integers such that

$$\sigma^*(fk) = f\sigma^*(k).$$

The utility of this concept arises in that if $m = km'$ and $n = kn'$ are unitary amicable numbers with

$$(k, m'n') = 1 = (fk, m'n'),$$

then fkm' and fkn' are also unitary amicable numbers. McClung found sixteen generators which he applied to the unitary amicable pairs in the Hagis list [1] to produce 25 unitary amicable pairs of which 3 are new.

In Section 1, properties of generators are investigated. An equivalence relation is defined on the set of generators. A product of two generators is defined, but not everywhere, which is consistent with the equivalence relation and so yields a product of classes, also not everywhere defined.

Section 2 is devoted to methods of producing generators. The action of classes of generators on unitary amicable pairs is defined and the properties are examined. The section closes with a table of generators.

Section 3 briefly indicates how the methods of Section 2 apply to unitary sociable sets, defined in [2].

H. J. J. te Riele, [5], [6], used number pairs (a, b), satisfying

$$\sigma^*(a)/a = \sigma^*(b)/b,$$

to generate hundreds of new unitary amicable pairs. One can define a binary operation and an equivalence relation on the set of all such pairs which yield stuctures isomorphic to those developed here for generators. Both te Riele [7] and McClung [4] were aware of the equivalence of the two methods. Apparently, neither developed the structures of the te Riele pairs to the extent this paper does for the McClung generators.

A paper in progress will extend and generalize this one.

1. Properties and Operations

It is assumed that the reader is familiar with McClung's results and notation [3]. To avoid confusion, $\gcd(a, b)$ wil denote the greater common divisor of a and b. $\mathrm{gucd}(a, b)$ will denote the greatest unitary common divisor. The notation (f, k) will be reserved for generators.

A prime p divides f if it divides either the numerator or the denominator of f. The expression "p is (not) in f" means that "p does (not) divide f." m is said to be relatively prime to f, i.e., $\gcd(m, f) = 1$, if no prime p divides both m and f.

If a prime p is relatively prime to fk but not to k, then it divides f.

Extend the definition of a generator as follows.

Definition 1: A pair of integers of the form $(1, k)$ will be called a *trivial generator*.

When applied to a given unitary amicable pair, a trivial generator does not generate a different unitary amicable pair. The two integers of a trivial generator are relatively prime. So on eliminating extraneous primes, one gets $(1, 1)$.

Theorem 1: a. (f, k) is a generator iff $(1/f, fk)$ is a generator.

b. Let p be a prime which does not divide f. Then, (f, kp^a) is a generator for all positive integers a.

Proof: a. f is a rational not one and k and fk are integers.

$$\sigma^*(fk) = f\sigma^*(k) \text{ iff } \sigma^*(k) = (1/f)\sigma^*(fk)$$
$$\text{iff } \sigma^*((1/f)(fk)) = (1/f)\sigma^*(fk).$$

$1/f$ is a rational not one and fk and $(1/f)(fk) = k$ are both integers. $(1/f, fk)$ is a generator. The argument is reversible.

b. If $p \nmid k$,

$$\sigma^*(fkp^a) = \sigma^*(fk)\sigma^*(p^a) = f\sigma^*(k)\sigma^*(p^a) = f\sigma^*(kp^a).$$

If $p \mid k$, set $k = k'p^b$, $\gcd(k', p) = 1$. By McClung's Lemma 2, (f, k') is a generator. By the previous case,

$$(f, kp^a) = (f, k'p^{a+b})$$

is a generator. □

Compare with McClung's Lemma 2. In effect, for any prime p which does not divide f, one can divide or multiply k by any power of p that yields an integer and thereby produce a new generator.

Since there are countably infinitely many primes p and prime powers p^a, each generator (f, k) has countably infinitely many generators (f, kp^a), $p \nmid f$, associated to it.

Definition 2: For (f, k), the generator $(1/f, fk)$ is called the *inverse* or *reciprocal generator* and is written $(f, k)^{-1}$.

Note that $(1, k)^{-1} = (1, k)$ and that $(1/f, fk)^{-1} = (f, k)$. Trivial generators are their own inverses. The inverse of the inverse is the initial generator.

Definition 3: A generator (f_1, k_1) is said to be *related* to a generator (f_2, k_2) iff (a) $f_1 = f_2$; and (b) there exist integers m and n both relatively prime to f_1, so that $mk_1 = nk_2$.

Theorem 2: The relation of Definition 3 is an equivalence relation.

Proof: Obvious. □

Definition 4: A generator (f, k) is said to be *primitive* if there does not exist a prime p, $p^a \| k$, $p \nmid f$, such that (f, kp^{-a}) is a generator.

Essentially, a generator is primitive iff k has no extraneous primes. Several properties are immediate consequences of Definition 3, Theorem 2, and Definition 4.

Trivial generators form an equivalence class.

Each generator (f, k) has a unique primitive generator associated to it by eliminating extraneous primes.

Each equivalence class has one and only one primitive generator.

For a primitive generator, if $p|k$, then $p|f$. Thus, $\pi(k) \le \pi(f)$.

$$(f, k) \text{ is primitive iff } (1/f, fk) = (f, k)^{-1} \text{ is primitive.}$$

Primitive generators are the natural representatives for the equivalence classes. Upper case letters (F, K) will be used for primitive generators. The primitive generator associated to an arbitrary generator (f, k) will be denoted by $\underline{(f, k)}$; the equivalence class, by $\langle(f, k)\rangle$. Arbitrary equivalence classes will be denoted by C_i.

McClung's conjectures can be stated in stronger form by using reciprocals and primitives. Even then, they are false.

Conjecture 1: Up to reciprocals, the only primitive generator (F, K) with $\pi(F) = \pi(K) = 2$ is $(3/2, 12)$.

$$\left(\frac{1}{2 \cdot 17}, 2^4 \cdot 17\right) \text{ is a counterexample.}$$

Conjecture 2: There are no primitive generators (F, K) with $\pi(F) > 2$ or $\pi(K) > 2$.

$$\left(\frac{2^3 \cdot 3 \cdot 5 \cdot 11 \cdot 43}{17}, 2^4 \cdot 3 \cdot 17\right) \text{ is a counterexample.}$$

Definition 5: Let (f_1, k_1) and (f_2, k_2) be two generators such that $k_2 = f_1 k_1$. Then the *product* of (f_1, k_1) and (f_2, k_2), in that order, is defined and given by

$$(f_1, k_1) \times (f_2, k_2) = (f_1 f_2, k_1).$$

Lemma 1: The product $(f_1 f_2, k_1)$ of two generators (f_1, k_1) and (f_2, k_2) is a generator. The product is trivial iff the factors are reciprocals.

Proof: k_1 and $f_1 f_2 k_1 = f_2 k_2$ are integers. We must show that

$$\sigma^*(f_1 f_2 k_1) = f_1 f_2 \sigma^*(k_1);$$

$$\sigma^*(f_1 f_2 k_1) = \sigma^*(f_2 k_2) = f_2 \sigma^*(k_2) = f_1 f_2 \sigma^*(k_1).$$

If the factors are reciprocals, the product is obviously trivial. Suppose the product is trivial. Then, $f_1 f_2 = 1$ and $f_2 = 1/f_1$. Substituting, the factors become (f_1, k_1) and $(1/f_1, f_1 k_1)$. □

From Definition 5, it is obvious that the product is not defined for every pair of generators. The product of $(2 \cdot 5, 2)$ and $(3 \cdot 41, 3^3)$ does not exist in either order. When the product does exist, it need not be commutative.

$$(2 \cdot 17, 2^3) \times \left(\frac{2 \cdot 11}{17}, 2^4 \cdot 17\right) = (2^2 \cdot 11, 2^3)$$

but is not defined in the opposite order.

For a generator (f, k), the trivial generator $(1, k)$ is a left identity, and $(1, fk)$ is a right identity.

Where sufficiently defined, the product is associative. Let (f_i, k_i), $i = 1, 2, 3$ be generators such that the products $(f_1, k_1) \times (f_2, k_2)$ and $(f_2, k_2) \times (f_3, k_3)$ exist. Then the product of the three is associative; i.e.,

$$((f_1, k_1) \times (f_2, k_2)) \times (f_3, k_3) = (f_1, k_1) \times ((f_2, k_2) \times (f_3, k_3)).$$

It is a simple matter to follow both sides through to $(f_1f_2f_3, k_1)$. The conditions necessary for each intermediate step obtain.

The reciprocal of a product is the product of the reciprocals in the reverse order.

$$((f_1, k_1) \times (f_2, k_2))^{-1} = (f_1f_2, k_1)^{-1} = (1/f_1f_2, f_1f_2k_1);$$
$$(f_2, k_2)^{-1} \times (f_1, k_1)^{-1} = (1/f_2, f_2k_2) \times (1/f_1, f_1k_1)$$
$$= (1/f_1f_2, f_2k_2) = (1/f_1f_2, f_1f_2k_1).$$

The equality $f_1k_1 = (1/f_2)(f_2k_2)$ must hold for the product of the reciprocals to exist. But $f_1k_1 = k_2 = (1/f_2)(f_2k_2)$. The product is defined!

Lemma 2: Let (f_i, k_i), $i = 1, 2, 3, 4$, be generators such that:

a. (f_1, k_1) is equivalent to (f_2, k_2); (f_3, k_3) to (f_4, k_4); and

b. the products $(f_1, k_1) \times (f_3, k_3)$ and $(f_2, k_2) \times (f_4, k_4)$ exist. Then, the products are equivalent.

Proof: By Definition 3, $f_1 = f_2$ and $f_3 = f_4$, so $f_1f_3 = f_2f_4$. Also, there exist integers m and n, both relatively prime to f_1 so that $mk_1 = nk_2$, and p and q, both relatively prime to f_3, so that $pk_3 = qk_4$. Assume m and n are relatively prime and p and q also. Otherwise, divide out the gcd's. As $k_3 = f_1k_1$ and $k_4 = f_2k_2$, $pf_1k_1 = qf_2k_2$ and $pk_1 = qk_2$. $k_1 = (q/p)k_2$ and $m(q/p)k_2 = nk_2$. $mq = pn$. m must divide p. Say $p = am$. $mq = amn$ and $q = an$. Thus, a divides $\gcd(p, q) = 1$, $a = 1$, $q = n$, and $p = m$. m and n are then relatively prime to both f_1 and f_3, hence to f_1f_3, and satisfy the condition for equivalence. □

Definition 6: Let C_1 and C_2 be two equivalence classes such that for (f_1, k_1) in C_1 and for (f_2, k_2) in C_2, the product $(f_1, k_1) \times (f_2, k_2)$ exists. Then, we say that the *product* of the two classes C_1 and C_2, in that order, exists and is given by:

$$C_1 \times C_2 = \langle (f_1, k_1) \times (f_2, k_2) \rangle.$$

The product is not everywhere defined. Where it is defined, by Lemma 2, it is well defined. Where it is defined, it is not necessarily commutative. It does have some nice properties which we list in the following theorem. No proofs are given as they follow from the preceding discussion.

Theorem 3: a. The class of trivial generators is a two-sided identity.

b. Each class has a two-sided inverse, or reciprocal, given by

$$\langle (f, k) \rangle^{-1} = \langle (f, k)^{-1} \rangle.$$

c. The reciprocal of a product is the product of the reciprocals in the reverse order.

d. The product is associative; that is, let C_i, $i = 1, 2, 3$, be classes such that the products $C_1 \times C_2$ and $C_2 \times C_3$ exist, then

$$(C_1 \times C_2) \times C_3 = C_1 \times (C_2 \times C_3).$$

The reciprocal of a class C will be denoted by C^{-1}. $(C^{-1})^{-1} = C$.

To form products, class representatives cannot be chosen at random. Even primitive generators are not necessarily good choices. The product

$$(2 \cdot 5, 2) \times (3 \cdot 41, 3^3)$$

is not defined. Equivalent generators yield

$$(2 \cdot 5,\ 2 \cdot 3^3) \times (3 \cdot 41,\ 2^3 \cdot 3^3 \cdot 5) = (2 \cdot 3,\ 5 \cdot 41,\ 2 \cdot 3^3).$$

Thus

$$\langle(2 \cdot 5)\rangle \times \langle(3 \cdot 41,\ 3^3)\rangle = \langle(2 \cdot 3,\ 5 \cdot 41,\ 2 \cdot 3^3)\rangle.$$

Lemma 3: Let C_1, C_2 be classes such that for respective generators (f_1, k_1) and (f_2, k_2), f_1, f_2 have no primes in common. Then the product $C_1 \times C_2$ exists and is commutative.

Proof: Let (f_1, K_1), (f_2, K_2) be the corresponding primitives. Since f_1, f_2 have no primes in common, neither do f_1, K_2 nor f_2, K_1 nor K_1, K_2. Then the products, in both orders, can be defined using equivalent generators. Specifically,

$$(f_1,\ K_1K_2) \times (f_2,\ f_1K_1K_2) = (f_1f_2,\ K_1K_2) = (f_2,\ K_2K_1) \times (f_1,\ f_2K_1K_2).$$

Thus,

$$C_1 \times C_2 = C_2 \times C_1. \ \square$$

The converse of Lemma 3 is an open question. The following is given without proof.

Corollary 1: Let C_1, C_2 be two classes such that for the respective primitives (F_1, K_1), (F_2, K_2), F_1, F_2 have no primes in common. Then the product exists, is commutative, and is given by $\langle(F_1F_2,\ K_1K_2)\rangle$.

Except for the fact that the product is not everywhere defined, the set of classes would form a group. The product fails to exist in one significant case so that the properties of the product as described set bounds on the best possible situation.

Lemma 4: With the exception of the identity class, the square of a class does not exist.

Proof: It is a direct calculation to show that the square of the identity is the identity. Let (F, K) be the primitive for any nonidentity class, $F \neq 1$. For the product to be defined there must be integers m and n, relatively prime to F so that the product $(F, Km) \times (F, Kn)$ exists; that is, so that $FKm = Kn$, $Fm = n$. Since m and n are relatively prime to F, either $F = 1$ or $m = n$, which forces $F = 1$. $\square$

Lemma 4 also implies that, with the exception of the identity class, the powers of a class do not exist. The full characterization of which products exist (or do not exist) is an open question.

2. Generators and Unitary Amicable Pairs

There are at least three methods of producing generators. McClung found sixteen in a limited computer search. Briefly, he characterized generators with $\pi(f) = 2$ and $\pi(k) = 1$ and searched for generators of the forms

$$(2 \cdot p,\ 2^a),\ (2^2 \cdot p,\ 2^b),\ \text{and}\ (3 \cdot p,\ 3^c).$$

He found five, eight, and three, respectively. By the nature of the characterization, all are primitive.

The characterization of other generator forms remains a fertile area of endeavor. It appears, for example, that in the case $\pi(f) = \pi(k) = 2$, f is not an integer.

The examination of known unitary amicable pairs yields generators. Before discussing the method, a brief review will be useful to allow the introduction of notation.

Two numbers m and n form a unitary amicable pair if

$$\sigma^*(m) = \sigma^*(n) = m + n.$$

Let $T = \text{gucd}(m, n)$. Write $m = TM$ and $n = TN$. Assume, for convenience, that $M < N$. Notation for the unitary amicable pair m, n will be $U = (T; M, N)$.

The action of a generator (f, k) on U to produce a new unitary amicable pair U' takes the following form. If k is a unitary divisor of T such that

$$\gcd(fk, (T/k)MN) = 1 = \gcd(k, (T/k)MN),$$

then the unitary amicable pair produced is $U' = (fT; M, N)$.

Use right function notation:

$$(f, k) : (T; M, N) \mapsto (fT; M, N) \quad \text{and} \quad (T; M, N)(f, k) = (fT; M, N).$$

Lemma 5: Let (f_1, k_1) and (f_2, k_2) be two generators which act on the unitary amicable pair $(T; M, N)$ to produce the same unitary amicable pair U'. Then one has that (f_1, k_1) is equivalent to (f_2, k_2).

Proof: Since $(T; M, N)(f_1, k_1) = (f_1T; M, N)$ and $(T; M, N)(f_2, k_2) = (f_2T; M, N)$,

$$(f_1T; M, N) = (f_2T; M, N) \text{ and } f_1T = f_2T.$$

Thus, $f_1 = f_2$. k_1 and k_2 are unitary divisors of T. There exist numbers a, b, also unitary divisors of T so that $ak_1 = T = bk_2$.

$$\gcd(a, k_1) = 1 = \gcd(b, k_2).$$

We must show that a and b are relatively prime to f_1. Suppose p is a prime dividing both a and f_1. Since $a = (T/k_1)$ and $\gcd(f_1k_1, (T/K_1)MN) = 1$, p does not divide f_1k_1. Thus, p must occur to a negative power in f_1 and a positive in k_1. However, since $\gcd(k_1, (T/k_1)MN) = 1$, it does not. Thus, p does not divide f_1. So a, and similarly b, is relatively prime to f_1. Therefore, one has that (f_1, k_1) and (f_2, k_2) are equivalent. □

Definition 7: Two unitary amicable pairs

$$U_1 = (T_1; M_1, N_1) \quad \text{and} \quad U_2 = (T_2; M_2, N_2)$$

are said to be *in the same family* iff $M_1 = M_2$ and $N_1 = N_2$.

Lemma 6: The relation of being in the same family is an equivalence relation.

Proof: Left to the reader. □

Since the action of a generator class on a unitary amicable pair $(T; M, N)$ leaves M and N unchanged, the classes cycle pairs within the family. Lemma 5 leads to the following statement.

Definition 8: A generator class C is said to *act on a unitary amicable pair* U to yield another pair U' if there is a generator (f, k) in C such that

$$U(f, k) = U'.$$

Notation will be $C : U \mapsto U'$ or $UC = U'$.

If C is the identity class, $UC = U$ for any U. If $UC = U'$, $U'C^{-1} = U$.

Theorem 4: Let U and U' be unitary amicable pairs in the same family. Then there is a class C so that $UC = U'$.

Proof: Let $U = (T; M, N)$ and $U' = (T'; M, N)$. It suffices to find a generator (f, k) so that $U(f, k) = U'$. $C = \langle (f, k) \rangle$. Let $f = T'/T$ and $k = T$. If $T' = T$, (f, k) is a trivial generator with the desired action. Assume $T' \neq T$. k and fk are integers.

$$\gcd(k, MN) = \gcd(T, MN) = 1 \quad \text{and} \quad \gcd(fk, MN) = \gcd(T', MN) = 1.$$

To verify that $f\sigma^*(k) = \sigma^*(fk)$, note that the relation $\sigma^*(TM) - TM = TN$, yields

$$\sigma^*(T)/T = (M + N)/\sigma^*(M).$$

Thus,

$$\sigma^*(T)/T = \sigma^*(T')/T';$$

$$\begin{aligned} f\sigma^*(k) &= (T'/T)\sigma^*(T) = (T'/T')\sigma^*(T') = \sigma^*(T'') \\ &= \sigma^*((T'/T)T) = \sigma^*(fk). \end{aligned}$$

Finally,

$$U(f, k) = (T; M, N)(T'/T, T) = ((T'/T)T; M, N) = (T'; M, N) = U'. \ \square$$

Corollary: The cardinality of the set of classes is at least as large as the cardinality of the largest family of unitary amicable pairs.

Theorem 5: Let (f_1, k_1) and (f_2, k_2) be generators, and let U_1, U_2, and U_3 be unitary amicable pairs in the same family, satisfying:

1. The product $(f_1, k_1) \times (f_2, k_2)$ exists;
2. $U_1(f_1, k_1) = U_2$;
3. $U_2(f_2, k_2) = U_3$.

Then $U_1((f_1, k_1) \times (f_2, k_2)) = U_3$.

Proof: By (1) the product

$$(f_1, k_1) \times (f_2, k_2) = (f_1 f_2, k_1)$$

exists. The action $U_1(f_1 f_2, k_1)$ is defined if k_1 is a unitary divisor of T_1 as given in (2). It suffices to evaluate $f_1 f_2 T_1$. From (2) and (3),

$$f_1 f_2 T_1 = f_2(f_1 T_1) = f_2 T_2 = T_3.$$

Thus,

$$U_1(f_1 f_2, k_1) = (T_3; M, N) = U_3. \ \square$$

Let C_1, C_2 be classes, and let U_1, U_2, U_3 be unitary amicable pairs in the same family, satisfying:

1. The product $C_1 \times C_2$ is defined;
2. $U_1 C_1 = U_2$;
3. $U_2 C_2 = U_3$.

Then, $U_1(C_1 \times C_2) = U_3$.

A brief list of primitive generators is given in the table below. Sources include McClung's list [3] and the results of applying Theorem 4 to the unitary amicable pairs in Hagis [1]. Inverses and products are not listed. The pairs listed by Wall [8] were not examined for the generators arising there. A description of another method of forming generators can be found in [3] and

[5]. No effort was made to produce generators for the list given here. This list is intended to be typical, not inclusive.

Table of Primitive Generators

1.	$(2 \cdot 3^2, 2^2)$	15.	$(2^3 \cdot 3 \cdot 17, 2)$
2.	$(2 \cdot 5, 2)$	16.	$(2^4 \cdot 11 \cdot 43, 2^3)$
3.	$(2 \cdot 17, 2^3)$	17.	$(2 \cdot 3 \cdot 5 \cdot 41, 2 \cdot 3^3)$
4.	$(2 \cdot 257, 2^7)$	18.	$(2^6 \cdot 3 \cdot 11 \cdot 43, 2)$
5.	$(2 \cdot 65537, 2^{15})$	19.	$\left(\frac{2 \cdot 11}{17}, 2^4 \cdot 17\right)$
6.	$(2^2 \cdot 3, 2)$	20.	$\left(\frac{2 \cdot 5 \cdot 13}{3^2}, 2 \cdot 3^3 \cdot 5\right)$
7.	$(2^2 \cdot 11, 2^3)$	21.	$\left(\frac{2^3 \cdot 11 \cdot 43}{17}, 2^4 \cdot 17\right)$
8.	$(2^2 \cdot 43, 2^5)$	22.	$\left(\frac{2^3 \cdot 3 \cdot 5 \cdot 11 \cdot 43}{17}, 2^4 \cdot 3 \cdot 17\right)$
9.	$(2^2 \cdot 683, 2^9)$	23.	$\left(\frac{2^3 \cdot 2^3 \cdot 11 \cdot 41 \cdot 43}{17}, 2^4 \cdot 3^3 \cdot 17\right)$
10.	$(2^2 \cdot 2731, 2^{11})$	24.	$(3 \cdot 5, 3)$
11.	$(2^2 \cdot 43691, 2^{15})$	25.	$(3 \cdot 41, 3^3)$
12.	$(2^2 \cdot 173763, 2^{17})$	26.	$(3 \cdot 21523361, 3^{15})$
13.	$(2^2 \cdot 2796203, 2^{21})$	27.	$\left(\frac{3}{2}, 2^2 \cdot 3\right)$
14.	$(2^2 \cdot 3^2 \cdot 17, 2^2)$	28.	$\left(\frac{5 \cdot 13}{3}, 3^3 \cdot 5\right)$

3. Unitary Sociable Numbers

Lal, Tiller, and Summers [2] defined unitary sociable numbers as sets of numbers m_i, $i = 1, 2, \ldots, n$, so that

$$\sigma^*(m_i) - m_i = m_{i+1}, \text{ for } i = 1, 2, \ldots, n-1,$$

and

$$\sigma^*(m_n) - m_n = m_1.$$

Use the convention that m_1 is the smallest number in the set.

Unitary sociable sets are extensions of unitary amicable pairs. The notation for amicable pairs can also be extended. Given a unitary sociable set, let

$$T = \text{gucd}(m_1, m_2, \ldots, m_n)$$

and set $m_i = TM_i$, $i = 1, 2, \ldots, n$. Then, one can denote a unitary sociable set by the notation

$$S = (T; M_1, \ldots, M_n).$$

All the results of Section 2 are valid with U's replaced by S's.

The T values were calculated for all sets in [2]. No matches were found between these values and the generators given in the table. The production of new unitary sociable sets by the methods of this paper must await more extensive lists of generators.

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