AN ASYMPTOTIC FORMULA CONCERNING A GENERALIZED EULER FUNCTION

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1. Introduction

Harlan Stevens [8] introduced the following generalization of the Euler φ-function. Let $F = \{f_1(x), \ldots, f_k(x)\}$, $k \geq 1$, be a set of polynomials with integral coefficients and let A represent the set of all ordered k-tuples of integers $(a_1, \ldots, a_k)$ such that $0 \leq a_1, \ldots, a_k \leq n$. Then $\varphi_F(n)$ is the number of elements in A such that the g.c.d. $(f_1(a_1), \ldots, f_k(a_k)) = 1$. We have, for $n = \Pi_{j=1}^{r} p_j^{e_j}$,

$$\varphi_F(n) = n^k \cdot \prod_{j=1}^{r}\left(1 - \frac{N_{1j} \ldots N_{kj}}{p_j^k}\right)$$

where N_{ij} is the number of incongruent solutions of $f_i(x) \equiv 0 \pmod{p_j}$, see [8, Theorem 1].

This totient function is multiplicative and it is very general. As special cases, we obtain Jordan's well-known totient $J_k(n)$ [3, p. 147] for $f_1(x) = \cdots = f_k(x) = x$; the Euler totient function $\varphi(n) \equiv J_1(n)$; Schemmel's function $\phi_t(n)$ [7] for $k = 1$ and $f_1(x) = x(x + 1) \ldots (x + t - 1)$, $t \geq 1$; also the totients investigated by Nagell [5], Alder [1], and others (cf. [8]).

The aim of this paper is to establish an asymptotic formula for the summatory function of $\varphi_F(n)$ using elementary arguments and preserving the generality. We shall assume that each polynomial $f_i(x)$ has relatively prime coefficients, that is, for each

$$f_i(x) = a_{ir_i}x^{r_i} + a_{ir_i-1}x^{r_i-1} + \cdots + a_{i0}$$

the g.c.d. $(a_{ir_i}, a_{ir_i-1}, \ldots, a_{i0}) = 1$.

2. Prerequisites

We need the following result stated by Stevens [8].

Lemma 1:

$$\varphi_F(n) = \sum_{d|n} \mu(d)\Omega_F(d)\left(\frac{n}{d}\right)^k, \tag{1}$$

where μ is the Möbius function and $\Omega_F(n)$ is a completely multiplicative function defined as follows: $\Omega_F(1) = 1$ and, for $1 < n = \Pi_{j=1}^{r} p_j^{e_j}$,

$$\Omega_F(n) = \prod_{j=1}^{r} (N_{1j} \ldots N_{kj})^{e_j}.$$

Under the assumption mentioned in the Introduction, we now prove

Lemma 2:

$$|\mu(n)\Omega_F(n)| = O(n^{\varepsilon}) \text{ for all positive } \varepsilon. \tag{2}$$

Proof: Suppose the congruence

$$f_i(x) = a_{ir_i}x^{r_i} + a_{ir_i-1}x^{r_i-1} + \cdots + a_{i0} \equiv 0 \pmod{p_j}$$

is of degree s_{ij}, $0 \le s_{ij} \le r_i$, where

$$a_{is_{ij}} \not\equiv 0 \pmod{p_j}.$$

Then, as is well known (by Lagrange's theorem), the congruence

$$f_i(x) \equiv 0 \pmod{p_j}$$

has at most s_{ij} incongruent roots, where $s_{ij} \le r_i$ for all primes p_j; therefore, $N_{ij} \le r_i$ for all primes p_j and $N_{ij} \le 2 + \max_{1 \le i \le k} r_i = M$, $M > 1$, for all i and j.

Now, for $n = \Pi_{j=1}^{r} p_j^{e_j}$, $|\mu(n)\Omega_F(n)| = 0$ if j exists such that $e_j \ge 2$; otherwise,

$$|\mu(n)\Omega_F(n)| = \left|(-1)^r \cdot \prod_{j=1}^{r}(N_{1j} \ldots N_{kj})\right| \le (M^k)^r.$$

Hence, $|\mu(n)\Omega_F(n)| \le A^{\omega(n)}$ for all n, where $A = M^k > 1$.

On the other hand, one has

$$2^{\omega(n)} = 2^r \le \prod_{j=1}^{r}(e_j + 1) = d(n),$$

so $\omega(n) \le \log_2 A$, which implies

$$|\mu(n)\Omega_F(n)| \le A^{\log_2 d(n)}.$$

Further, it is known that $d(n) = O(n^{\alpha})$ for all $\alpha > 0$ (see [4, Theorem 315]). By choosing $\alpha = \varepsilon/\log_2 A > 0$, we obtain $|\mu(n)\Omega_F(n)| = O(n^{\varepsilon})$, as desired.

Lemma 3: The series

$$\sum_{n=1}^{\infty} \frac{\mu(n)\,\Omega_F(n)}{n^{s+1}}$$

is absolutely convergent for $s > 0$, and its sum is given by

$$\lambda_F(s) = \prod_p \left(1 - \frac{N_1 \cdots N_k}{p^{s+1}}\right), \tag{3}$$

where N_i denotes the number of incongruent solutions of $f_i(x) \equiv 0 \pmod{p}$.

Proof: The absolute convergence follows by Lemma 2:

$$|\mu(n)\Omega_F(n)/n^{s+1}| \le K \cdot 1/n^{s+1-\varepsilon},$$

where $K > 0$ is a constant and $\varepsilon > 0$ is such that $s - \varepsilon > 0$. Note that the general term is multiplicative in n, so the series can be expanded into an infinite Euler-type product [3, 17.4]:

$$\sum_{n=1}^{\infty} \frac{\mu(n)\Omega_F(n)}{n^s} = \prod_p \left(\sum_{\ell=0}^{\infty} \frac{\mu(p^{\ell})\Omega_F(p^{\ell})}{p^{\ell s}}\right) = \prod_p \left(1 - \frac{\Omega_F(p)}{p^s}\right) = \lambda_F.$$

From here on, we shall use the following well-known estimates.

Lemma 4:

$$\sum_{n \le x} n^s = \frac{x^{s+1}}{s+1} + O(x^s), \quad s > 1; \tag{4}$$

$$\sum_{n \le x} \frac{1}{n^s} = O(x^{1-s}), \quad 0 < s < 1; \tag{5}$$

$$\sum_{n > x} \frac{1}{n^s} = O\left(\frac{1}{x^{s-1}}\right), \quad s > 1. \tag{6}$$

3. Main Results

Theorem 1:

$$\sum_{n \le x} \varphi_F(n) = \frac{\lambda_F(k)x^{k+1}}{k+1} + O(R_k(x)), \tag{7}$$

where $R_k(x) = x^k$ or $x^{1+\varepsilon}$ (for all $\varepsilon > 0$) according as $k \ge 2$ or $k = 1$.

Proof: Using (1) and (4), one has

$$\begin{aligned}\sum_{n \le x} \varphi_F(n) &= \sum_{d\delta = n \le x} \mu(d)\Omega_F(d)\delta^k = \sum_{d \le x} \mu(d)\Omega_F(d) \sum_{\delta \le x/d} \delta^k \\ &= \sum_{d \le x} \Omega_F(d)\mu(d)\left\{\frac{1}{k+1} \cdot (x/d)^{k+1} + O((x/d)^k)\right\} \\ &= \frac{x^{k+1}}{k+1} \cdot \sum_{d=1}^{\infty} \frac{\mu(d)\Omega_F(d)}{d^{k+1}} + O\left(x^{k+1} \cdot \sum_{d > x} \frac{|\mu(d)\Omega_F(d)|}{d^{k+1}}\right) \\ &\quad + O\left(x^k \cdot \sum_{d \le x} \frac{|\mu(d)\Omega_F(d)|}{d^k}\right).\end{aligned}$$

Here the main term is

$$\frac{\lambda_F(k)x^{k+1}}{k+1}$$

by (3); then, in view of (2) and (6), the first remainder term becomes

$$O\left(x^{k+1} \cdot \sum_{d > x} \frac{d^{\varepsilon}}{d^{k+1}}\right) = O\left(x^{k+1} \cdot \sum_{d > x} \frac{1}{d^{k+1-\varepsilon}}\right) = O(x^{1+\varepsilon}) \quad \text{(choosing } 0 < \varepsilon < 1).$$

For the second remainder term, (2) implies

$$O\left(x^k \cdot \sum_{d \le x} \frac{d^{\varepsilon}}{d^k}\right) = O\left(x^k \cdot \sum_{d \le x} \frac{1}{d^{k-\varepsilon}}\right),$$

which is

$$O(x^k) \text{ for } k \ge 2, \text{ and } O(x \cdot x^{1-1+\varepsilon}) = O(x^{1+\varepsilon}) \text{ for } k = 1 \text{ [by (5)]}.$$

This completes the proof of the theorem.

For $f_1(x) = \cdots = f_k(x) = x$, we have $N_{ij} = 1$ for all i and j; thus, $\varphi_F(n) = J_k(n)$ - the Jordan totient function. This yields

Corollary 1 (cf. [2, (3.7) and (3.8)]):

$$\sum_{n \le x} J_k(n) = \frac{x^{k+1}}{(k+1)\zeta(k+1)} + O(x^k), \quad k \ge 2; \tag{8}$$

$$\sum_{n \le x} \varphi(n) = \frac{x^2}{2\zeta(2)} + O(x^{1+\varepsilon}), \quad k = 1, \text{ for all } \varepsilon > 0, \tag{9}$$

where $\zeta(s)$ is the Riemann zeta function.

Remark: The O-term of (9) can easily be improved into $O(x \log x)$, see Mertens' formula [4, Theorem 330].

By selecting $k = 1$ and $f_1(x) = x(x+1) \ldots (x+t-1)$, $t \geq 1$, we get

$\varphi_F(n) = \phi_t(n)$ - Schemmel's totient function [7],

for which $N_1 = p$ if $p < t$, and $N_1 = t$ if $p \geq t$. Using Theorem 1, we conclude

Corollary 2:

$$\sum_{n \leq x} \phi_t(n) = \frac{x^2}{2} \prod_{p<t}\left(1 - \frac{1}{p}\right) \cdot \prod_{p \geq t}\left(1 - \frac{t}{p^2}\right) + O(x^{1+\varepsilon}) \text{ for all } \varepsilon > 0. \tag{10}$$

For $t = 2$, $\phi_2(n) \equiv \varphi'(n)$, see [6, p. 37, Ex. 20], and we have

Corollary 3:

$$\sum_{n \leq x} \varphi'(n) = \frac{x^2}{2} \cdot \prod_{p}\left(1 - \frac{2}{p^2}\right) + O(x^{1+\varepsilon}) \text{ for all } \varepsilon > 0. \tag{11}$$

Choosing $k = 1$ and $f_1(x) = x(\lambda - x)$, we obtain

$\varphi_F(n) \equiv \theta(\lambda, n)$ - Nagell's totient function [5],

where $N_1 = 1$ or 2, according as $p \mid \lambda$ or $p \nmid \lambda$, and we have

Corollary 4:

$$\sum_{n \leq x} \theta(\lambda, n) = \frac{x^2}{2} \cdot \prod_{p \mid \lambda}\left(1 - \frac{1}{p^2}\right) \cdot \prod_{p \nmid \lambda}\left(1 - \frac{2}{p^2}\right) + O(x^{1+\varepsilon}) \text{ for all } \varepsilon > 0. \tag{12}$$

Now, let $f_1(x) = \cdots = f_k(x) = x^2 + 1$, $N_i = 1, 2$, or 0, according as $p = 2$, $p \equiv 1 \pmod 4$, or $p \equiv 3 \pmod 4$, see [8, Ex. 4]. In this case, we have

Corollary 5:

$$\sum_{n \leq x} \varphi_F(n) = \frac{x^{k+1}}{k+1}\left(1 - \frac{1}{2^{k+1}}\right) \cdot \prod_{p \equiv 1 \pmod 4}\left(1 - \frac{2^k}{p^{k+1}}\right) \tag{13}$$

$$+ O(R_k(x)), \text{ with } R_k(x) \text{ as given in Theorem 1.}$$

Theorem 2: Let $f(x)$ be a polynomial with integral coefficients. The probability that for two positive integers a, b, $a \leq b$, we have $(f(a), b) = 1$ is

$$\prod_{p}\left(1 - \frac{N(p)}{p^2}\right),$$

where $N(p)$ denotes the number of incongruent solutions of $f(x) \equiv 0 \pmod p$.

Proof: Let n be a fixed positive integer and consider all the pairs of integers (a, b) satisfying $1 \leq a \leq b \leq n$:

$$\begin{matrix} (1,1) & (1,2) & (1,3) & \ldots & (1,n) \\ & (2,2) & (2,3) & \ldots & (2,n) \\ & & (3,3) & \ldots & (3,n) \\ & & & \ddots & \vdots \\ & & & & (n,n) \end{matrix}$$

There are

$$A(n) = \frac{n(n+1)}{2} \sim \frac{n^2}{2}$$

such pairs and the property $(f(a), b) = 1$ is true for $B(n)$ pairs of them, where

$$B(n) = \varphi_F(1) + \varphi_F(2) + \cdots + \varphi_F(n) \sim \frac{n^2}{2} \cdot \prod_p \left(1 - \frac{N(p)}{p^2}\right) \text{ by Theorem 1.}$$

Hence, the considered probability is

$$\lim_{n\to\infty} \frac{B(n)}{A(n)} = \prod_p \left(1 - \frac{N(p)}{p^2}\right).$$

As immediate consequences, we obtain, for example:

Corollary 6 [4, Theorem 332]: The probability of two positive integers being prime to one another is

$$1/\zeta(2) = 6/\pi^2.$$

Corollary 7 ($\Omega_F(n) = \phi_2(n)$): The probability that, for two positive integers a and b, $a \leq b$, we have $(a(a+1), b) = 1$, is

$$\prod_p \left(1 - \frac{2}{p^2}\right).$$

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