ON THE F-REPRESENTATION OF INTEGRAL SEQUENCES $\{F_n^2/d\}$ AND $\{L_n^2/d\}$ WHERE d IS EITHER A FIBONACCI OR A LUCAS NUMBER

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1. Introduction

A new look at Zeckendorf's theorem [1] has led to several seemingly unexpected results [2], [3], [4], [5], [6]. It is the purpose of this paper to extend previous findings [4] by involving squares of Fibonacci numbers (F_n) and Lucas numbers (L_n).

The Fibonacci representation of a positive integer N (F-representation of N) [1] is defined to be the representation of N as a sum of positive, distinct, nonconsecutive Fibonacci numbers. It is unique [7]. The number of terms in this representation is symbolized by $f(N)$.

Consider the sequences

$$\{F_n^2/F_m\}, \quad \{F_n^2/L_m\}, \quad \{L_n^2/L_m\}, \quad \{L_n^2/F_m\}.$$

Necessary interrelationships between n and m need to be stipulated to assure integral elements in these sequences. We will predict the number of terms (F-addends) necessary in these representations, and will also exhibit the representations themselves.

Beyond the identities I_7, I_{14} - I_{18}, and I_{21} - I_{24} available in [7], the following further identities are used in the proofs of theorems:

$$\sum_{i=1}^{r} F_{ai+b} = \frac{F_{a(r+1)+b} + (-1)^{a-1}F_{ar+b} - F_{a+b} + (-1)^a F_b}{L_a + (-1)^{a-1} - 1}; \tag{1.1}$$

$$L_{n+k} - (-1)^k L_{n-k} = 5F_nF_k; \tag{1.2}$$

$$L_{n+k} + (-1)L_{n-k} = L_nL_k. \tag{1.3}$$

Their validity can be readily proved with the aid of the Binet form for F_n and L_n. In particular, (1.1) plays a prominent role throughout the proofs.

2. The F-Representation of F_{sk}^2/F_s

If s is an odd positive integer and k is a natural number, then

$$f(F_{sk}^2/F_s) = \begin{cases} sk/2 & \text{if } k \text{ is even} \\ s(k-1)/2 + 1 & \text{if } k \text{ is odd} \end{cases} \tag{2.1}$$

and

$$F_{sk}^2/F_s = \begin{cases} \displaystyle\sum_{i=1}^{k/2} \sum_{j=1}^{s} F_{4si+2j-3s-1} & \text{if } k \text{ is even,} \quad (2.2) \\ \displaystyle F_s + \sum_{i=1}^{(k-1)/2} \sum_{j=1}^{s} F_{4si+2j-s-1} & \text{if } k \text{ is odd.} \quad (2.3) \end{cases}$$

Proof of (2.2) (k is even): Using (1.1), I_{23}, I_{24}, I_{16}, and I_7, the right-hand side of (2.2) can be rewritten as

$$\begin{aligned} &\sum_{i=1}^{k/2} (F_{4si-s+1} - F_{4si-s-1} - F_{4si-3s+1} + F_{4si-3s-1}) \\ &= \sum_{i=1}^{k/2} (F_{4si-s} - F_{4si-3s}) = L_s \sum_{i=1}^{k/2} F_{4si-2s} \\ &= L_s(F_{2sk+2s} - F_{2sk-2s} - 2F_{2s})/(L_{4s} - 2) \\ &= L_s(L_{2sk}F_{2s} - 2F_{2s})/(L_{4s} - 2) = L_sF_{2s}(L_{2sk} - 2)/(L_{4s} - 2) \\ &= 5L_sF_{2s}F_{sk}^2/(5F_{2s}^2) = L_sF_{sk}^2/F_{2s} = F_{sk}^2/F_s. \end{aligned}$$

Proof of (2.3) (k is odd): Using (1.1), I_{23}, I_{24}, I_{16}, I_7, and I_{17}, the right-hand side of (2.3) can be rewritten as

$$\begin{aligned} &F_s + \sum_{i=1}^{(k-1)/2} (F_{4si+s+1} - F_{4si+s-1} - F_{4si-s+1} + F_{4si-s-1}) \\ &= F_s + \sum_{i=1}^{(k-1)/2} (F_{4si+s} - F_{4si-s}) = F_s + L_s \sum_{i=1}^{(k-1)/2} F_{4si} \\ &= F_s + L_s(F_{2sk+2s} - F_{2sk-2s} - F_{4s})/(L_{4s} - 2) \\ &= F_s + L_s(L_{2sk}F_{2s} - F_{4s})/(L_{4s} - 2) \\ &= F_s + F_{2s}L_s(L_{2sk} - L_{2s})/(5F_{2s}^2) = F_s + L_s(L_{2sk} - L_{2s})/(5F_{2s}) \\ &= (5F_sF_{2s} + L_s(L_{2sk} - L_{2s}))/(5F_{2s}) = L_s(5F_s^2 + L_{2sk} - L_{2s})/(5F_{2s}) \\ &= (5F_s^2 + L_{2sk} - L_{2s})/(5F_s) = (L_{2sk} + 2)/(5F_s) \\ &= 5F_{sk}^2/(5F_s) = F_{sk}^2/F_s. \end{aligned}$$

(2.1) follows readily from (2.2) and (2.3).

As a particular case, letting $s = 1$ in (2.1), (2.2), and (2.3), we have (see also [3])

$$f(F_k^2) = [(k+1)/2], \tag{2.4}$$

where $[x]$ denotes the greatest integer not exceeding x, and

$$F_k^2 = \begin{cases} \displaystyle\sum_{j=1}^{k/2} F_{4j-2} & \text{if } k \text{ is even,} \\ \displaystyle F_2 + \sum_{j=1}^{(k-1)/2} F_{4j} & \text{if } k \text{ is odd.} \end{cases} \tag{2.5}$$

Theorem 2: If s is an even positive integer and k is a natural number, then

$$f(F_{sk}^2/F_s) = k \tag{2.6}$$

and

$$F_{sk}^2/F_s = \sum_{j=1}^{k} F_{2sj-s}. \tag{2.7}$$

Proof of (2.7): Using (1.1), I_{24}, and I_{16}, the right-hand side of (2.7) can be rewritten as

$$\begin{aligned}(F_{2sk+s} - F_{2sk-s} - 2F_s)/(L_{2s} - 2) &= (L_{2sk}F_s - 2F_s)/L_{2s} - 2)\\ &= (F_s(L_{2sk} - 2))/(L_{2s} - 2) = (5F_sF_{sk}^2)/(5F_s^2) = F_{sk}^2/F_s.\end{aligned}$$

(2.6) is an immediate consequence of (2.7).

As a particular case, letting $s = 2$ in (2.6) and (2.7), we have [cf. (2.4) and (2.5)]

$$f(F_{2k}^2) = k \tag{2.8}$$

and

$$F_{2k}^2 = \sum_{j=1}^{k} F_{4j-2}. \tag{2.9}$$

3. The F-Representation of F_{tk}^2/L_s

Theorem 3: ($t = 2s$) If s is an odd positive integer and k is a natural number, then

$$f(F_{2sk}^2/L_s) = (s+1)k/2 \tag{3.1}$$

and

$$F_{2sk}^2/L_s = \sum_{i=1}^{k}\left(F_{4si-3s+1} + \sum_{j=1}^{(s-1)/2} F_{4si+4j-3s}\right). \tag{3.2}$$

Proof of (3.2): Using (1.1), I_{14}, (1.2), I_{24}, I_{16}, and I_7, the right-hand side of (3.2) can be rewritten as

$$\begin{aligned}&\sum_{i=1}^{k}(F_{4si-3s+1} + (F_{4si-s+2} - F_{4si-s-2} - F_{4si-3s+4} + F_{4si-3s})/5)\\ &= \sum_{i=1}^{k}(L_{4si-s} - F_{4si-3s+4} + F_{4si-3s} + 5F_{4si-3s+1})/5\\ &= \sum_{i=1}^{k}(L_{4si-s} + L_{4si-3s})/5 = F_s\sum_{i=1}^{k}F_{4si-2s}\\ &= F_s(F_{4sk+2s} - F_{4sk-2s} - 2F_{2s})/(L_{4s} - 2)\\ &= F_s(L_{4sk}F_{2s} - 2F_{2s})/(L_{4s} - 2) = F_sF_{2s}(L_{4sk} - 2)/(L_{4s} - 2)\\ &= 5F_sF_{2s}F_{2sk}^2/(5F_{2s}^2) = F_sF_{2sk}^2/F_{2s} = F_{2sk}^2/L_s.\end{aligned}$$

(3.1) follows.

Theorem 4: ($t = 2s$) If s is an even positive integer and k is a natural number, then

$$f(F_{2sk}^2/L_s) = sk/2 \tag{3.3}$$

and

$$F_{2sk}^2/L_s = \sum_{i=1}^{k}\sum_{j=1}^{s/2} F_{4si+4j-3s-2}. \tag{3.4}$$

Proof of (3.4): As in the proof of Theorem 3, using (1.1), I_{14}, (1.2), I_{24}, I_{16}, and I_7, the right-hand side of (3.4) can be rewritten as

$$\sum_{i=1}^{k}(F_{4si-s+2} - F_{4si-s-2} - F_{4si-3s+2} + F_{4si-3s-2})/5$$

$$= \sum_{i=1}^{k}(L_{4si-s} - L_{4si-3s})/5 = F_s\sum_{i=1}^{k}F_{4si-2s} = F_{2sk}^2/L_s.$$

(3.3) follows.

It can be noted that, by letting $s = 2$ in (3.3) and (3.4), we obtain the same identities as those resulting from $s = 4$ in (2.6) and (2.7).

Theorem 5: $(t = s = 3)$ If k is an odd positive integer, then

$$f(F_{3k}^2/L_3) = k \tag{3.5}$$

and

$$F_{3k}/4 = F_2 + \sum_{j=1}^{(k-1)/2}(F_{12j-2} + F_{12j+1}). \tag{3.6}$$

Proof of (3.6): Using (1.1), I_{24}, and I_{17}, the right-hand side of (3.6) can be rewritten as

$$F_2 + 2\sum_{j=1}^{(k-1)/2}F_{12j} = F_2 + 2(F_{6k+6} - F_{6k-6} - F_{12})/(L_{12} - 2)$$

$$= 1 + (8L_{6k} - 144)/160 = (L_{6k} + 2)/20 = 5F_{3k}^2/20 = F_{3k}^2/4.$$

(3.5) results from (3.6).

4. The F-Representation of L_{sk}^2/L_s

Theorem 6: $(s = 1)$ If k is a natural number, then

$$f(L_k^2/L_1) = f(L_k^2) = \begin{cases} k & \text{if } k = 1, 2 \\ 3 & \text{if } k \ge 4 \text{ is even} \\ k - 1 & \text{if } k \ge 3 \text{ is odd} \end{cases} \tag{4.1}$$

and

$$L_k^2/L_1 = L_k^2 = \begin{cases} F_3 + F_{2k-1} + F_{2k+1} & \text{if } k \ge 4 \text{ is even,} \quad (4.2) \\ F_{2k+1} + \sum_{j=1}^{k-2}F_{2j+2} & \text{if } k \ge 3 \text{ is odd.} \quad (4.3) \end{cases}$$

Proof of (4.2): $(k \ge 4$ is even) Using I_{15}, the right-hand side of (4.2), which is given by the sum of three F-addends, can be rewritten as

$$F_3 + L_{2k} = L_{2k} + 2 = L_k^2.$$

Proof of (4.3): $(k \ge 3$ is odd) Using (1.1) and I_{18}, the right-hand side of (4.3) can be rewritten as

$$F_{2k+1} + F_{2k} - F_{2k-2} - 2 = L_{2k} - 2 = L_k^2.$$

(4.1) follows as it is trivial for $k = 1$ or 2.

Theorem 7: If s and k are odd positive integers $(s > 1)$, then

$$f(L_{sk}^2/L_s) = 2k \tag{4.4}$$

and

$$L_{sk}^2/L_s = \sum_{j=1}^{k}(F_{2sj-s-1} + F_{2sj-s+1}). \tag{4.5}$$

Proof of (4.5): Using (1.1), (1.3), and I_{18}, the right-hand side of (4.5) can be rewritten as

$$\begin{aligned}(L_{2sk+s} - L_{2sk-s} - 2L_s)/(L_{2s} - 2) &= (L_s L_{2sk} - 2L_s)/(L_{2s} - 2)\\ = L_s(L_{2sk} - 2)/(L_{2s} - 2) &= L_s L_{sk}^2/L_s^2 = L_{sk}^2/L_s.\end{aligned}$$

(4.4) follows.

Theorem 8: If s is an even positive integer and k is an odd positive integer, then

$$f(L_{sk}^2/L_s) = \begin{cases} 1 & \text{if } k = 1 \\ k+1 & \text{if } k \geq 3 \end{cases} \text{ and } s = 2 \quad \begin{matrix} \\ \\ (s+1)(k-1)/2 + 2 \quad \text{if } s \geq 4 \end{matrix} \tag{4.6}$$

and (for $s = 2$):

$$L_{2k}^2/3 = \begin{cases} F_4 & \text{if } k = 1 \\ F_2 + F_{4k-1} + \sum_{j=1}^{(k-1)/2} (F_{8j-3} + F_{8j-1}) & \text{if } k \geq 3; \end{cases} \tag{4.7}$$

(for $s \geq 4$):

$$L_{sk}^2/L_s = F_{s-1} + F_{s+1} + \sum_{i=1}^{(k-1)/2}\Big(F_{4si-s-2} + F_{4si-s+1} + F_{4si+s+1} + \sum_{j=1}^{s-2} F_{4si+2j-s+2}\Big). \tag{4.8}$$

Proof of (4.7): ($s = 2$) The statement clearly holds for $k = 1$. For $k \geq 3$, using (1.1), (1.2), and I_{15}, the right-hand side of (4.7) can be rewritten as

$$\begin{aligned}&F_{2k} + F_{4k-1} + (L_{4k+2} - L_{4k-6} - 15)/(L_8 - 2)\\ &= 1 + F_{4k-1} + (5F_4F_{4k-2} - 15)/45 = 1 + F_{4k-1} + (F_{4k-2} - 1)/3\\ &= 3F_{4k-1} + F_{4k-2} + 2)/3 = (L_{4k} + 2)/3 = L_{2k}^2/3.\end{aligned}$$

Proof of (4.8): ($s \geq 4$) Using (1.1), I_{14}, (1.2), I_{24}, I_7, I_{16}, and I_{15}, the right-hand side of (4.8) can be rewritten as

$$\begin{aligned}&L_s + \sum_{i=1}^{(k-1)/2} (F_{4si-s-2} + F_{4si-s+1} + F_{4si+s+1} + F_{4si+s}\\ &\qquad - F_{4si+s-2} - F_{4si-s+4} + F_{4si-s+2})\\ &= L_s + \sum_{i=1}^{(k-1)/2} (F_{4si-s-2} - F_{4si-s+2} + F_{4si+s+2} - F_{4si+s-2})\\ &= L_s + \sum_{i=1}^{(k-1)/2} (L_{4si+s} - L_{4si-s}) = L_s + 5F_s \sum_{i=1}^{(k-1)/2} F_{4si}\\ &= L_s + 5F_s(F_{2sk+2s} - F_{2sk-2s} - F_{4s})/(L_{4s} - 2)\\ &= L_s + 5F_sF_{2s}(L_{2sk} - L_{2s})/(5F_{2s}^2) = L_s + F_s(L_{2sk} - L_{2s})/F_{2s}\\ &= L_s + (L_{2sk} - L_{2s})/L_s = (L_s^2 + L_{2sk} - L_{2s})/L_s\\ &= (L_{2sk} + 2)/L_s = L_{sk}^2/L_s.\end{aligned}$$

(4.6) follows from (4.7) and (4.8).

5. The F-Representation of L_n^2/F_m for Certain Values of n and m

Theorem 9: ($n = 3k$, $m = 3$) If k is a natural number, then

$$f(L_{3k}^2/F_3) = 2k - 1 \tag{5.1}$$

and

$$L_{3k}^2/2 = \begin{cases} F_5 + F_{6k} + \sum_{j=1}^{2k-3} F_{3j+4} & \text{if } k \text{ is even} \quad (5.2) \\ F_{6k} + \sum_{j=1}^{2k-2} F_{3j+1} & \text{if } k \text{ is odd.} \quad (5.3) \end{cases}$$

Proof of (5.2) (k is even): Using (1.1), I_{23}, I_{22}, and I_{15}, the right-hand side of (5.2) can be rewritten as

$$\begin{aligned} &F_5 + F_{6k} + (F_{6k-2} + F_{6k-5} - F_7 - F_4)/L_3 \\ &= 5 + (4F_{6k} + F_{6k-2} + F_{6k-5} - 16)/14 \\ &= 1 + (F_{6k+3} - F_{6k-3} + F_{6k-2} + F_{6k-5})/4 \\ &= 1 + (F_{6k+3} + F_{6k-3})/4 \\ &= 1 + L_{6k}F_3/4 = 1 + L_{6k}/2 = (L_{6k} + 2)/2 = L_{3k}^2/2. \end{aligned}$$

Proof of (5.3) (k is odd): As before, using I_{18} instead of I_{15}, the right-hand side of (5.3) becomes

$$\begin{aligned} &F_{6k} + (F_{6k-2} + F_{6k-s} - F_4 - F_1)/L_3 \\ &= (4F_{6k} + F_{6k-2} + F_{6k-s})/4 - 1 \\ &= L_{6k}/2 - 1 = (L_{6k} - 2)/2 = L_{3k}/2. \end{aligned}$$

(5.1) follows from (5.2) and (5.3), regardless of the parity of k.

Theorem 10: ($n = 6k - 3$, $m = 6$) If k is a natural number, then

$$f(L_{6k-3}^2/F_6) = 3k - 2 \tag{5.4}$$

and

$$L_{6k-3}^2/8 = F_3 + \sum_{j=1}^{k-1} (F_{12j-4} + F_{12j-1} + F_{12j+3}). \tag{5.5}$$

Proof of (5.5): Using I_{21}, (1.1), I_{24}, I_{22}, and I_{18}, the right-hand side of (5.5) can be rewritten as

$$\begin{aligned} &F_3 + \sum_{j=1}^{k-1} (F_{12j-4} + 3F_{12j+1}) \\ &= F_3 + (F_{12k-4} - F_{12k-16} + 3(F_{12k+1} - F_{12k-11}) - 720)/320 \\ &= (F_6 L_{12k-10} + 3F_6 L_{12k-5} - 80)/320 = (3L_{12k-5} + L_{12k-10} - 10)/40 \\ &= (F_{12k-1} + F_{12k-11} - 10)/40 = (5L_{12k-6} - 10)/40 \\ &= (L_{6(2k-1)} - 2)/8 = L_{3(2k-1)}^2/8. \end{aligned}$$

(5.4) follows.

Note that the case ($n = 2k$, $m = 4$) is nothing but the case (4.7) of Theorem 8.

6. Concluding Remarks

F-representations of the sequences

$$\{F_{sk}^2/F_s\},\ \{F_{2sk}^2/L_s\},\ \{F_{3k}^2/L_3\},\ \{L_{sk}^2/L_s\},\text{ and }\{L_n^2/F_m\}$$

have been investigated and their f-functions, as well as the specific summation expressions, have been given. The authors believe that the results presented in this paper are new. Many further analogous sequences could be analyzed. Possibly, some of the work above could be extended to simple cases of the sequences $\{F_n^k/d\}$ and $\{L_n^k/d\}$, where $k \geq 1$, and d is a power of certain Fibonacci or Lucas numbers. The authors hope to continue their investigations in this area. As an example, we offer the sequences:

$$\text{(i)}\quad \{F_{sk}^2/F_s^2\}\quad\text{and}\quad\text{(ii)}\quad \{F_{skF_s}^2/F_s^2\}.$$

Example (i): ($s = 4$)

$$f(F_{4k}^2/F_4^2) = \begin{cases} (4k-1)/3 & \text{if } k \equiv 1 \pmod 3\\ (4k+1)/3 & \text{if } k \equiv 2 \pmod 3\\ 4k/3 & \text{if } k \equiv 0 \pmod 3\end{cases} \tag{6.1}$$

and

$$F_{4k}^2/9 = \begin{cases} F_2 + \sum_{i=1}^{(k-1)/3}(F_{24i-10} + F_{24i-5} + F_{24i-1} + F_{24i+1}) & \text{if } k \equiv 1 \pmod 3,\\ F_3 + F_7 + F_9 + \sum_{i=1}^{(k-2)/3}(F_{24i-2} + F_{24i+3} + F_{24i+7} + F_{24i+9}) & \text{if } k \equiv 2 \pmod 3,\\ \sum_{i=1}^{k/3}(F_{24i-18} + F_{24i-13} + F_{24i-9} + F_{24i-7}) & \text{if } k \equiv 0 \pmod 3.\end{cases} \tag{6.2}$$

Example (ii): ($s = 4$) It can be proved that, for $s > 2$, $F_s^2 | F_m$ if and only if $m = ksF_s$ ($k = 0, 1, \ldots$). In this particular case, we have (cf. [4], Th. 5).

$$f(F_{4kF_4}/F_4^2) = f(F_{12k}/9) = \begin{cases} 3k & \text{if } k \text{ is even}\\ 3k-1 & \text{if } k \text{ is odd}\end{cases} \tag{6.3}$$

and

$$F_{12k}/9 = \begin{cases} \sum_{i=1}^{k/2}(F_{24i-19} + F_{24i-5} + \sum_{j=1}^{4} F_{24i+2j-17}) & \text{if } k \text{ is even}\\ F_4 + F_7 + \sum_{i=1}^{(k-1)/2}(F_{24i-7} + F_{24i+7} + \sum_{j=1}^{4} F_{24i+2j-5}) & \text{if } k \text{ is odd.}\end{cases} \tag{6.4}$$

We leave the proofs of these illustrative examples to the enjoyment of the reader.

References

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3. P. Filipponi. "The Representation of Certain Integers as a Sum of Distinct Fibonacci Numbers." Int. Rept. 2B0985, Fondazione Ugo Bordoni, Roma, 1985.

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