TRAPPING A REAL NUMBER BETWEEN ADJACENT RATIONALS

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Introduction

In this article we ask the following question: Given any real number σ can one find a rational number p/q such that $(p+1)/(q+1) < \sigma < p/q$.? Clearly, one of the necessary conditions of this problem is that $\sigma > 1$. But this condition is not sufficient. Interestingly enough, the question came up as a result in algebraic geometry in [2], where Sommese essentially proves the sufficiency of $\sigma > 2$ in the first theorem.

We give explicit conditions under which the above question is true using a somewhat stronger hypothesis: Given any real number $\sigma > 1$ and $N > 0$, can one find positive integers r and s such that $r > s > N$, and s divisible by some fixed integer m, and the denominator of a *fixed* rational number t and satisfying $r - ts > M$, *for any* M, where

$$1 < t < \sigma \quad \text{and} \quad \frac{r+1}{s+1} < \sigma < \frac{r}{s}?$$

The answer depends on whether σ is rational or irrational. We have the following two theorems:

Theorem 1: Let $\sigma = p/q$ be a positive rational number. Then the following are equivalent:

i) $\sigma > 2$

ii) Given any positive integers m, M, N and a rational number $t = a/b$ such that $0 < t < \sigma$, then one can find r and s such that $r > s > N$, s is divisible by mb, and

$$r - ts > M \quad \text{and} \quad \frac{r+1}{s+1} < \sigma < \frac{r}{s}.$$

Proof: First we prove that ii) $\Rightarrow$ i). Since mb divides s, write $s = nmb$, where n is a positive integer. Since $r - ts > M$, we must have $r = ts + M + u$ for some integer $u \geq 1$. Hence, $r = nma + M + u$. Thus,

$$\begin{aligned}\frac{p}{q} > \frac{r+1}{s+1} &\Rightarrow p(s+1) > q(r+1)\\ &\Rightarrow sp + p - q > qr\\ &\quad\; nmbp + p - q > qnma + qM + qu\\ &\Rightarrow p - q(u+1) > qnma - nmbp + qM\\ &\Rightarrow \frac{p - q(u+1)}{nqb} > m\left(\frac{a}{b} - \frac{p}{q}\right) + \frac{M}{nb}.\end{aligned}$$

Now choose M sufficiently large so that

$$m\left(\frac{a}{b} - \frac{p}{q}\right) + \frac{M}{nb} \geq 0.$$

Hence, we conclude that

$$\frac{p - q(u + 1)}{nqb} > 0.$$

Thus, $p > q(u + 1)$, from which it follows that

$$\sigma = \frac{p}{q} > 2.$$

Next we show that i) $\Rightarrow$ ii). Let $r = np + 1$, $s = nq$. Choose $n = kmb$. Then $r - ts = n(p - tq) + 1 \to \infty$, as $k \to \infty$, since $p - tq > 0$. It is also easily seen that

$$\sigma > 2 \Rightarrow \frac{r + 1}{s + 1} < \sigma. \quad \square$$

Before discussing the next theorem, we need a few results.

Let $a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$ denote a continued fraction.

Let $a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = \frac{p_n}{q_n}$, then $p_n = Q(a_0, a_1, \ldots, a_n)$ and $q_n = Q(a_1, \ldots, a_n)$.

Unlike [1], we use $Q(a_0, a_1, \ldots, a_n)$ to denote Euler continuants, where each of p_n and q_n are expanded using Euler's rule ([1], p. 82). Also well known is that (see [1], p. 83),

$$Q(a_0, \ldots, a_n) = a_0 Q(a_1, \ldots, a_n) + Q(a_2, \ldots, a_n). \tag{*}$$

Remark 1: By Euler's rule, as $n \to \infty$, $p_n \to \infty$, $q_n \to \infty$, and $Q(a_2, \ldots, a_n) \to \infty$.

$$p_n - q_n = (a_0 - 1)q_n + Q(a_2, \ldots, a_n), \text{ by } (*).$$

We also know (see [1], p. 84) that

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}. \tag{**}$$

Let α be an irrational number.

Let $a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{1}{\alpha_{n+1}} = \alpha$.

Then $\alpha_{n+1} > 1$, and is irrational. Moreover (see [1], p. 89),

$$\alpha = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}}.$$

And, by (**), it follows that, if *n is even*, then

$$\frac{p_n}{q_n} < \alpha < \frac{p_{n-1}}{q_{n-1}}. \tag{***}$$

This brings us to Theorem 2.

Theorem 2: Suppose that σ is irrational, and $\sigma > 1$. Let $t = a/b$ be a fixed rational number and m a fixed positive integer. Given any $N > 0$, one can find positive integers r and s, with $r > s > N$, s is divisible by mb, satisfying $r - ts > M$, for any given M, where

$$0 < t < \frac{r + 1}{s + 1} < \sigma < \frac{r}{s}.$$

Proof: Let σ have a continued fraction representation as α above. By (***), we see that, for n even,

$$\frac{p_n}{q_n} < \alpha < \frac{p_{n-1}}{q_{n-1}}.$$

Let $r = mabMp_{n-1}$ and $s = mabMq_{n-1}$, then

$$r - ts = maM(bp_{n-1} - aq_{n-1}) > M, \text{ since } \frac{a}{b} < \sigma < \frac{p_{n-1}}{q_{n-1}}.$$

$$\frac{p_n}{q_n} - \frac{r+1}{s+1} = \frac{p_n - q_n - mabM}{(mabMq_{n-1} + 1)q_n} > 0 \text{ if } n >> 0, \text{ and } n \text{ is even.}$$

This follows from (**) and Remark 1 above, noting that m, a, b, and M are given and n is arbitrary. Also

$$br - as = mabM(bp_{n-1} - aq_{n-1}) > a - b.$$

The last inequality holds since $bp_{n-1} - aq_{n-1} \geq 1$ and $ab > a - b$; hence,

$$t < \frac{r+1}{s+1}.$$

This proves the theorem. □

Example 1: The following example shows that if the conditions in part ii) of Theorem 1 are relaxed, then the implication is false. Let $\sigma = 8/5$, $r = 5$, and $s = 3$, then $6/4 < \sigma < 5/3$.

Example 2: If $\sigma = (n + 1)/n$, then it is easy to see that it is impossible to find r and s in Theorem 1, even under relaxed conditions. If $\sigma = p/q$, a careful examination of the proof shows that $p - q \geq 2$ is a necessary condition.

Remark 2: If $\sigma = 2$, then we can easily see that, for any $r/s > 2$, we must have $(r + 1)/(s + 1) \geq 2$. Hence, Theorem 1 fails in that case even in the relaxed form.

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References

1. Harold Davenport. *The Higher Arithmetic.* London: Hutchinson University Library, 1970.
2. A. J. Sommese. "On the Adjunction Theoretic Structure of Projective Varieties; Proceedings of Conference in Gottingen, 1985. Springer Lecture Notes #1194.

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