# UNITARY PERFECT NUMBERS WITH SQUAREFREE ODD PART

S. W. GRAHAM
Michigan Technological University, Houghton, MI 49931
(Submitted July 1987)

## 1. Introduction

A divisor $d$ of a natural number $n$ is said to be *unitary* if and only if

$$(d, n/d) = 1.$$

The sum of the unitary divisors of $n$ is denoted $\sigma^*(n)$. It is straightforward to show that if

$$n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k},$$

then

$$\sigma^*(n) = (p_1^{a_1} + 1)(p_2^{a_2} + 1) \cdots (p_k^{a_k} + 1).$$

A natural number $n$ is said to be *unitary perfect* if $\sigma^*(n) = 2n$.

Subbarao and Warren [2] discovered the first four unitary perfect numbers:

$$6 = 2 \cdot 3,\; 60 = 2^2 \cdot 3 \cdot 5,\; 90 = 2 \cdot 3^2 \cdot 5,\; 87360 = 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13.$$

Wall [3] discovered another such number,

$$146361946186458562560000 = 2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313,$$

and he later showed [4] that this is the fifth unitary perfect number. No other unitary perfect numbers are known, and Wall [5] has shown that any other such number must have an odd prime divisor exceeding $2^{15}$.

In this paper, we consider the existence of unitary perfect numbers of the form $2^m s$, where $s$ is a squarefree odd integer. We shall prove that there are only three such numbers.

*Theorem:* If $2^m s$ is a unitary perfect number and $s$ is squarefree, then either $m = 1$ and $s = 3$, $m = 2$ and $s = 3 \cdot 5$, or $m = 6$ and $s = 3 \cdot 5 \cdot 7 \cdot 13$.

## 2. Preliminaries

Throughout this paper, the letter $s$ shall be used to denote an odd squarefree number. The letter $p$, with or without a subscript, shall denote an odd prime. The letter $q$, with or without a subscript, shall denote a Mersenne prime.

Our starting point is the observation that, for any fixed $m$, it is easy to determine all unitary perfect numbers of the form $2^m s$. From the previously stated formula for $\sigma^*(n)$, we see that if $s = p_1 p_2 \ldots p_r$, then $2^m s$ is unitary perfect if and only if

$$2 = \frac{\sigma^*(2^m s)}{2^m s} = \frac{2^m + 1}{2^m} \cdot \frac{p_1 + 1}{p_1} \cdot \frac{p_2 + 2}{p_2} \cdot \ldots \cdot \frac{p_r + 1}{p_r}. \tag{1}$$

Any odd prime dividing $2^m + 1$ must appear as a denominator on the right-hand side. If $p$ is such a prime, then all odd prime divisors of $p + 1$ must also appear as denominators on the right-hand side. If we can force a prime to appear more than once, then we can conclude that there is no unitary perfect number of the form $2^m s$.

For example, suppose $m = 7$. Since $2^7 + 1 = 3 \cdot 43$, 3 and 43 must appear as denominators on the right-hand side of (1). Since $11 \mid (43 + 1)$, 11 must also appear. But $3 \mid (11 + 1)$, so 3 must appear twice. Therefore, there is no unitary perfect number of the form $2^7 s$.

On the other hand, suppose $m = 6$. Since $2^6 + 1 = 5 \cdot 13$, both 5 and 13 must be prime divisors of $s$. Since $3 \mid (5 + 1)$ and $7 \mid (13 + 1)$, 7 and 13 must be prime divisors of $s$. If any other $p$ divides $s$, then

$$\frac{\sigma^*(2^m s)}{2^m s} \geq \frac{2^6 + 1}{2^6} \cdot \frac{3 + 1}{3} \cdot \frac{5 + 1}{5} \cdot \frac{7 + 1}{7} \cdot \frac{13 + 1}{13} \cdot \frac{p + 1}{p} > 2.$$

Therefore, the only unitary perfect number of the form $2^6 s$ is $2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$.

Proceeding in this fashion, it is easy to show that the only unitary perfect numbers of the form $2^m s$ with $m < 10$ are those listed in the theorem. Thus, we may assume henceforth that $m \geq 10$. (Alternatively, we could reduce to the case $m \geq 10$ by quoting a result of Subbarao [1].)

The method of the preceding paragraphs is "top-down": we start with divisors of $2^m + 1$ and work down. While this procedure works well for specific $m$, it does not lend itself well to a proof in the general case. We therefore introduce an alternative "bottom-up" procedure. This procedure starts with the Mersenne primes dividing $s$ and works up to the divisors of $2^m + 1$. (A Mersenne prime is a prime of the form $2^k - 1$; the first few such primes are

$$3 = 2^2 - 1,\ 7 = 2^3 - 1,\ 31 = 2^5 - 1,\ 127 = 2^7 - 1,\ 8191 = 2^{13} - 1.)$$

First we note that $s$ does have Mersenne prime divisors. For in equation (1), all odd prime divisors of $\sigma^*(s) = p_1 p_2 \ldots p_r$ must appear in the denominator of the right-hand side. But some of the $p_i$'s divide $2^m + 1$, so at least one of the terms $p_i + 1$ must be free of any odd prime factors. It follows that $p_i$ is a Mersenne prime.

Suppose $q$ is a Mersenne prime dividing $s$. Renumber the primes in (1) so that $q = p_1$. There is some (necessarily unique) prime $p_2$ dividing $s$ such that $p_1 \mid (p_2 + 1)$. Note that $p_2 \geq 2p_1 - 1$. Either $p_2 \mid (2^m + 1)$ or there is some $p_3$ such that $p_2 \mid (p_3 + 1)$. Continuing in this way, we obtain a sequence of primes

$$p_1 < p_2 < \cdots < p_k, \tag{2}$$

where $p_1$ is a Mersenne prime, $p_k \mid (2^m + 1)$, and $p_{i+1} \geq 2p_i - 1$.

To formalize the ideas of the preceding paragraph, we introduce the following function $f$. Let $p$ be an odd prime in the denominator of the right-hand side of (1). We define $f(p)$ to be 1 if $p \mid (2^m + 1)$. Otherwise, we define $f(p)$ to be the unique prime $p'$ such that $p' \mid s$ and $p \mid (p' + 1)$. We define

$$f_0(p) = p,\ f_1(p) = f(p),\ \text{and}\ f_{k+1}(p) = f(f_k(p)).$$

We also define

$$f(1) = 1 \quad \text{and} \quad f_\infty(p) = \prod_{i=0}^{\infty} f_i(p).$$

For example, if $m = 6$ and $s = 3 \cdot 5 \cdot 7 \cdot 13$, then

$$f_1(3) = 5,\ f_2(3) = 1,\ \text{and}\ f_\infty(3) = 3 \cdot 5.$$

Similarly,

$$f_\infty(7) = 7 \cdot 13.$$

Let $q_1, q_2, \ldots, q_\ell$ be the Mersenne primes dividing $s$. Then all odd primes dividing $s$ occur in the product

$$f_\infty(q_1)f_\infty(q_2) \cdots f_\infty(q_\ell). \tag{3}$$

At this point, we cannot rule out the possibility that this product contains repeated prime factors. For example, if $41 \mid s$, then $41 \mid f_\infty(3)$ and $41 \mid f_\infty(7)$. Accordingly, for each Mersenne prime $q$ , we define $F(q_i)$ to be the product of all primes that divide $f_\infty(q_i)$ but do not divide any of $f_\infty(q_1), f_\infty(q_2), \ldots, f_\infty(q_{i-1})$. With this definition, we have

$$\frac{2^m + 1}{2^m} \cdot \frac{\sigma^*(F(q_1))}{F(q_1)} \cdot \ldots \cdot \frac{\sigma^*(F(q_\ell))}{F(q_\ell)} = 2.$$

If we write

$$G(q) = \frac{\sigma^*(F(q))}{F(q)},$$

then the above may be rewritten as

$$\frac{2^m + 1}{2^m} G(q_1) \cdots G(q_\ell) = 2. \tag{4}$$

The idea behind the proof is to obtain upper bounds for $G(q)$ that make (4) untenable. The crucial point here is that, if $p_1, p_2, \ldots, p_k$ are the primes described in (2), then $p_2 \geq 2p_1$, $p_3 \geq 4p_1 - 3$, etc. It follows that

$$G(q) \leq \prod_{i=0}^{\infty} \frac{2^i p_i - 2^i + 2}{2^i p_i - 2^i + 1}.$$

As we shall show in Lemmas 1 and 2, this product converges. This bound for $G$ is sufficient for the larger Mersenne primes. A more elaborate analysis is needed for the smaller primes.

## 3. Lemmas

*Lemma 1:* If $\rho$ and $\delta$ are real numbers with $\rho > 1$, then

$$\prod_{i=0}^{\infty} \frac{\delta\rho^i - (\rho + \rho^2 + \cdots + \rho^i)}{\delta\rho^i - (1 + \rho + \rho^2 + \cdots + \rho^i)} = \frac{(\rho - 1)\delta}{(\rho - 1)\delta - \rho}.$$

*Proof:* The $K^{\text{th}}$ partial product is

$$\frac{\delta}{\delta - 1} \cdot \frac{\rho\delta - \rho}{\rho\delta - \rho - 1} \cdot \ldots \cdot \frac{\rho^K\delta - \rho^K - \cdots - \rho}{\rho^K\delta - \rho^K - \cdots - \rho - 1}.$$

Note that the numerator of each term after the first is $\rho$ times the denominator of the previous term. Therefore, the $K^{\text{th}}$ partial product is

$$\frac{\delta\rho^K}{\rho^K\delta - \rho^K - \cdots - \rho - 1} = \frac{(\rho - 1)\delta}{(\rho - 1)\delta - \rho + \rho^{-K}}.$$

The result follows by letting $K$ tend to infinity.

*Lemma 2:* If $q = 2^m - 1$ is a Mersenne prime, then $G(q) \leq \dfrac{2^{m-1}}{2^{m-1} - 1}$.

*Proof:* Let $p_1, p_2, \ldots, p_k$ be the primes dividing $G(q)$. Since $p_1 = q = 2^m$ and and $p_{i+1} \geq 2p_i$, we see that

$$p_i \geq 2^{m+i-1} - 2^i + 1.$$

Therefore,

$$G(q) \leq \frac{2^m}{2^m - 1} \cdot \frac{2^{m+1} - 2}{2^{m+1} - 3} \cdots .$$

The result now follows by applying Lemma 1 with $\delta = 2^m$ and $\rho = 2$.

*Lemma 3:* Let $q_j, \ldots, q_\ell$ be the Mersenne primes that divide $s$ and are at least 8191. Then

$$G(q_j) \cdots G(q_\ell) \leq \frac{3072}{3071}.$$

*Proof:* It is well known that, if $2^m - 1$ is prime, then $m$ must be prime. Thus, $m = 2$ or $m$ is odd. Consequently

$$G(q_j) \cdots G(q_\ell) \leq \prod_{i=0}^{\infty} \frac{2^{12+2i}}{2^{12+1i} - 1}$$

We bound this by observing that

$$\prod_{i=0}^{\infty} \frac{2^{12+2i}}{2^{12+2i} - 1} \leq \prod_{i=0}^{\infty} \frac{2^{12+2i} - 4 - 4^2 - \cdots - 4^i}{2^{12+2i} - 1 - 4 - 4^2 - \cdots - 4^i}.$$

The result now follows from Lemma 1 with $\delta = 2^{12}$ and $\rho = 4$.

*Lemma 4:* Let $q_j, \ldots, q_\ell$ be the Mersenne primes that divide $s$ and are at least 127. Then

$$G(q_j) \cdots G(q_\ell) \leq \frac{122}{121}.$$

*Proof:* We first get a bound on $G(127)$. Let $p_1, \ldots, p_r$ be the primes that divide $F(127)$. If $r \leq 1$, then $G(127) \leq 128/127$. Assume that $r \geq 2$. Then $p_1 = 127$ and $p_2$ is a prime of the form $127h - 1$, where all the odd prime divisors of $h$ are at least 8191. Now $127 \cdot 2^i - 1$ is composite for $1 \leq i \leq 7$, so $p_2 \geq 127 \cdot 2^8 - 1 = 32511$. Therefore,

$$G(127) \leq \frac{128}{127} \prod_{i=0}^{\infty} \frac{32511 \cdot 2^i - 2 - 2^2 - \cdots - 2^i}{32511 \cdot 2^i - 1 - 2 - \cdots - 2^i} = \frac{128}{127} \cdot \frac{16256}{16255}.$$

From this and Lemma 3, we see that

$$G(q_j) \cdots G(q_\ell) \leq G(127)G(8191) \cdots \leq \frac{128}{127} \cdot \frac{16256}{16255} \cdot \frac{3072}{3071} \leq \frac{122}{121}.$$

## 4. Proof of the Theorem

As stated in Section 2, we may assume that $m \geq 10$.

The proof breaks into three cases: (1) $m$ odd, (2) $m \equiv 0 \bmod 4$, and (3) $m \equiv 2 \bmod 4$.

*Case 1:* Assume that $m$ is odd. Then $3 \mid 2^m + 1$, and $G(3) = 4/3$. It follows that the left-hand side of (4) is

$$\frac{2^m + 1}{2^m} \frac{4}{3} G(7)G(31) \cdots \leq \frac{1025}{1024} \frac{4}{3} \frac{4}{3} \frac{16}{15} \frac{122}{121} < 2.$$

*Case 2:* Assume that $m \equiv 0 \bmod 4$. Then $2^m + 1 \equiv 2 \bmod 3$ and $2^m + 1 \equiv 2 \bmod 5$. It follows that there is some prime $p$ such that $p \mid 2^m + 1$, $p \equiv 2 \bmod 3$, and $p > 5$. Moreover, the congruence $x^4 \equiv -1 \bmod p$ has the solution $x \equiv 2^{m/4}$, so we have $p \equiv 1 \bmod 8$. By the Chinese Remainder Theorem, $p \equiv 17 \bmod 24$. We cannot have $p = 17$ since $3^2 \mid \sigma^*(17)$. Therefore, $p \geq 41$, and the left-hand side of (4) is

$$\frac{2^m + 1}{2^m}\,\frac{4}{3}\,\frac{p+1}{p}\,G(7)G(31)\,\cdots \leq \frac{1025}{1024}\,\frac{4}{3}\,\frac{42}{41}\,\frac{4}{3}\,\frac{16}{15}\,\frac{122}{121} < 2.$$

*Case 3:* Assume that $m \equiv 2 \bmod 4$. Then $5 \mid 2^m + 1$, and

$$G(3) = \frac{4}{3}\,\frac{6}{5}.$$

This case breaks into four subcases: (i) $7 \nmid s$; (ii) $7 \mid s$ and $13 \nmid s$; (iii) $7 \mid s$, $13 \mid s$, and $103 \nmid s$; (iv) $7 \mid s$, $13 \mid s$, and $103 \mid s$.

*Subcase 3(i):* Assume that $7 \nmid s$. Then the left-hand side of (4) is

$$\frac{2^m + 1}{2^m}\,G(3)G(31)G(127)\,\cdots \leq \frac{1025}{1024}\,\frac{4}{3}\,\frac{6}{5}\,\frac{16}{15}\,\frac{122}{121} < 2.$$

*Subcase 3(ii):* Assume that $7 \mid s$ and $13 \nmid s$. Other than 13, the least prime of the form $7h - 1$ with all odd prime divisors of $h$ greater than or equal to 31 is $7 \cdot 32 - 1 = 223$. Therefore,

$$G(7) \leq \frac{8}{7}\prod_{i=0}^{\infty}\frac{224 \cdot 2^i - (2 + 2^2 + \cdots + 2^i)}{224 \cdot 2^i - (1 + 2 + \cdots + 2^i)} = \frac{8}{7}\,\frac{112}{111}.$$

Therefore, the left-hand side of (4) is

$$\leq \frac{1025}{1024}\,\frac{4}{3}\,\frac{6}{5}\,\frac{8}{7}\,\frac{112}{111}\,\frac{16}{15}\,\frac{122}{121} < 2.$$

*Subcase 3(iii):* Assume that $7 \mid s$, $13 \mid s$, and $103 \mid s$. Then $31 \mid s$ since

$$\frac{\sigma^*(3 \cdot 5 \cdot 7 \cdot 13 \cdot 31)}{3 \cdot 5 \cdot 7 \cdot 13 \cdot 31} > 2.$$

If $F(7)$ contains any prime factors other than 7 or 13, then the least such factor is of the form $13h - 1$, where all odd prime factors of $h$ are $\geq 127$. Other than 103, the least prime of this form is $13 \cdot 2^7 - 1 = 1663$. Therefore,

$$G(7) \leq \frac{8}{7}\,\frac{14}{13}\,\frac{832}{831},$$

and the left-hand side of (4) is

$$\leq \frac{1025}{1024}\,\frac{4}{3}\,\frac{6}{5}\,\frac{8}{7}\,\frac{14}{13}\,\frac{832}{831}\,\frac{122}{121} < 2.$$

*Subcase 3(iv):* Assume that $7 \mid s$, $13 \mid s$, and $103 \mid s$. Then $127 \nmid s$ since

$$\frac{\sigma^*(3 \cdot 5 \cdot 7 \cdot 13 \cdot 103 \cdot 127)}{3 \cdot 5 \cdot 7 \cdot 13 \cdot 103 \cdot 127} > 2$$

The least prime of the form $103h - 1$ is $103 \cdot 8 - 1 = 823$. Therefore,

$$G(7) \leq \frac{8}{7}\,\frac{14}{13}\,\frac{104}{103}\,\frac{412}{411},$$

and the right-hand side of (4) is

$$\leq \frac{1025}{1024}\,\frac{4}{3}\,\frac{6}{5}\,\frac{8}{7}\,\frac{14}{13}\,\frac{104}{103}\,\frac{412}{411}\,\frac{3072}{3071} < 2.$$

## References

1. M. V. Subbarao. "Are There an Infinity of Unitary Perfect Numbers?" *Amer. Math. Monthly* 77 (1970):389-390.
2. M. V. Subbarao & L. J. Warran. "Unitary Perfect Numbers." *Canad. Math. Bull.* 9 (1966):147-153; MR 33 #3994.
3. C. R. Wall. "A New Unitary Perfect Number." *Notices Amer. Math. Soc.* 16 (1969):825.
4. C. R. Wall. "The Fifth Unitary Perfect Number." *Canad. Math. Bull.* 18 (1975):115-122; MR 51 #12690.
5. C. R. Wall. "On the Largest Odd Component of a Unitary Perfect Number." *Fibonacci Quarterly* 25.4 (1987):312-316.

*****