ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-435 *Proposed by Ratko Tošić, University of Novi Sad, Yugoslavia*

(a) Prove that, for $n \geq 1$,

$$F_{n+1} + \sum_{\substack{0 < i_1 < \cdots < i_k \leq n \\ 1 \leq k \leq n}} F_{n+1-i_k} F_{i_k - i_{k-1}} \cdots F_{i_2 - i_1} F_{i_1} = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n+1}{2k+1} \cdot 2^k,$$

where $\lfloor x \rfloor$ is the greatest integer $\leq x$.

(b) Prove that, for $n \geq 3$,

$$\sum_{\substack{0 < i_1 < \cdots < i_k \leq n \\ 1 \leq k \leq n}} (-1)^{n-k} F_{n-1-i_k} F_{i_k - i_{k-1}} \cdots F_{i_2 - i_1} F_{i_1 - 2} \cdot 2^k = F_{n+3} + (-1)^{n+1} F_{n-3}.$$

(Comment: The identity is valid for $n \geq 0$, if we define
$F_{-3} = 2$, $F_{-2} = -1$; $F_i = F_{i-1} + F_{i-2}$, for $i \geq -1$.)

H-436 *Proposed by Piero Filipponi, Rome, Italy*

For p an arbitrary prime number, it is known that

$$(p-1)! \equiv p-1 \pmod{p}, \quad (p-2)! \equiv 1 \pmod{p},$$

and

$$(p-3)! \equiv (p-1)/2 \pmod{p}.$$

Let k_0 be the smallest value of an integer k for which $k! > p$.

The numerical evidence turning out from computer experiments suggests that the probability that, for k varying within the interval $[k_0, p-3]$, $k!$ reduced modulo p is either even or odd is 1/2. Can this conjecture be proved?

SOLUTIONS

Integrate Your Results

H-410 *Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 25, no. 2, May 1987)*

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \text{ for } n \geq 2.$$

Prove or disprove that, for $n \geq 1$,

$$\int_0^1 F_n(x)dx = \frac{1}{n}(L_n - (-1)^n - 1).$$

Solution by Paul Bruckman, (formerly) Fair Oaks, CA

The conjecture is true.

Proof: The characteristic equation of the $F_n(x)$ is given by:

(1) $$z^2 - xz - 1 = 0,$$

which has solutions

(2) $$u \equiv u(x) = \tfrac{1}{2}(x + \theta),\ v \equiv v(x) = \tfrac{1}{2}(x - \theta),$$

where $\theta \equiv \theta(x) = (x^2 + 4)^{\frac{1}{2}} = u - v$.

From the initial conditions on the $F_n(x)$, we readily find:

(3) $$F_n(x) = \frac{u^n - v^n}{u - v},\ n = 0, 1, 2, \ldots .$$

Also, we define $L_n(x)$ as follows:

(4) $$L_n(x) = u^n + v^n,\ n = 0, 1, 2, \ldots .$$

We may differentiate the quantities in (2) with respect to x; denoting such derivatives by prime symbols, we readily obtain:

(5) $$\theta' = x/\theta;\quad u' = u/\theta;\quad v' = -v/\theta.$$

From (4) and (5), we find:

$$L_n'(x) = nu^{n-1} \cdot u/\theta - nv^{n-1} \cdot v/\theta = n(u^n - v^n)/\theta,$$

or

(6) $$L_n'(x) = nF_n(x).$$

It follows from (6) that

$$\int_0^1 F_n(x)dx = L_n(x)/n\Big]_0^1, \text{ or}$$

(7) $$\int_0^1 F_n(x)dx = \frac{1}{n}(L_n(1) - L_n(0)),\ n = 1, 2, \ldots .$$

Now $\theta(1) = 5^{\frac{1}{2}}$, so $u(1) = \alpha$, $v(1) = \beta$ (the usual Fibonacci constants), and $L_n(1) = L_n$. Also, $\theta(0) = 2$, so $u(0) = 1$, $v(0) = -1$, and $L(0) = 1 + (-1)^n$. Thus,

(8) $$\int_0^1 F_n(x)dx = \frac{1}{n}(L_n - 1 - (-1)^n),\ n = 1, 2, \ldots . \quad \text{Q.E.D.}$$

Also solved by O. Brugia & P. Filipponi, C. Georghiou, L. Kuipers, J.-Z. Lee & J.-S. Lee, B. Prielipp, and the proposer.

Close Ranks

H-411 *Proposed by Paul S. Bruckman, Fair Oaks, CA*
(Vol. 25, no. 2, May 1987)

Define the simple continued fraction $\theta(a, d)$ as follows:

$$\theta(a, d) \equiv [a, a + d, a + 2d, a + 3d, \ldots], \ a \text{ and } d \text{ real}, \ d \neq 0.$$

Find a closed form for $\theta(a, d)$.

Solution by C. Georghiou, University of Patras, Patras, Greece

Take the differential equation

$$(*) \qquad zw'' + bw' - w = 0.$$

Then, for $b \neq 0, -1, -2, \ldots$, we have

$$\frac{w}{w'} = b + \frac{z}{w'/w''}.$$

By differentiating $(*)$, we get

$$\frac{w'}{w''} = b + 1 + \frac{z}{w''/w'''},$$

and by repeated differentiation of $(*)$, we get the continued fraction

$$(**) \qquad f(z) = \frac{w}{w'} = b + \cfrac{z}{b + 1 + \cfrac{z}{b + 2 + \cfrac{z}{b + 3 + \cdots}}}$$

Now it is shown in W. B. Jones & W. J. Thron, *Continued Fractions* (New York: Addison-Wesley, 1980), pp. 209-210, that the above continued fraction converges to the meromorphic function

$$f(z) = \frac{b_0F_1(b;\ z)}{{}_0F_1(b+1;\ z)}$$

for all complex numbers z and, moreover, the convergence is uniform on every compact subset of $\mathbb{C}$ that contains no poles of $f(z)$.

From the theory of continued fractions, we know that

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots = b_0 + \frac{c_1a_1}{c_1b_1} + \frac{c_1c_2a_2}{c_2b_2} + \frac{c_2c_3a_3}{c_3b_3} + \cdots$$

Where $c_n \neq 0$. Then, if we take $b = a/d$ and $z = 1/d^2$ in $(**)$ and use the above identity, we find

$$\theta(a, d) = \frac{a_0F_1(a/d);\ 1/d^2)}{{}_0F_1(a/d + 1;\ 1/d^2)}$$

valid for $a/d \neq 0, -1, -2, \ldots$ and $d \neq 0$. Since

$${}_0F_1(b + 1; \tfrac{1}{4}z^2) = e^{-z}M(b + \tfrac{1}{2}, 2b + 1, 2z) = \Gamma(b + 1)(\tfrac{1}{2}z)^{-b}I_b(z)$$

where $M(a, b, z)$ and $I_b(z)$ are the Confluent Hypergeometric function and the Modified Bessel function of the first kind, respectively, $\theta(a, d)$ can be expressed in terms of these functions as

$$\theta(a,\ d) = \frac{I_{a/d-1}(2/d)}{I_{a/d}(2/d)} = \frac{aM(a/d - \frac{1}{2},\ 2a/d - 1,\ 4/d)}{M(a/d + \frac{1}{2},\ 2a/d + 1,\ 4/d)}.$$

When $a/d = -k$, $k = 0, 1, 2, \ldots$, we have

$$\theta(-kd,\ d) = -kd + \frac{1}{\theta(-(k-1)d,\ d)}$$

and since

$$\theta(0,\ d) = \frac{1}{\theta(d,\ d)} = \frac{I_1(2/d)}{I_0(2/d)} = \frac{I_{-1}(2/d)}{I_0(2/d)},$$

it is easily shown by induction that

$$\theta(-kd,\ d) = \frac{I_{-k-1}(2/d)}{I_{-k}(2/d)}.$$

Therefore,

$$\theta(a,\ d) = \frac{I_{a/d-1}(2/d)}{I_{a/d}(2/d)}$$

for all (complex) a and d, $d \neq 0$ and $I_{a/d}(2/d)$ does not vanish. Since the Modified Bessel functions have no real zeros, the above expression is valid for all real a and d, $d \neq 0$.

Finally, for (real) a and $d = 0$, we have the simple periodic continued fraction

$$\theta(a,\ 0) = (a + \sqrt{a^2 + 4})/2 \text{ for } a > 0,$$

$\theta(-a,\ 0) = -\theta(a,\ 0)$, and $\theta(0,\ 0)$ diverges.

Also solved by the proposer who noted the following interesting result:

(*) $\quad \theta(1,\ 2) = \coth 1.$

It Adds Up!

H-412 *Proposed by Andreas N. Philippou & Frosso S. Makri, University of Patras, Patras, Greece (Vol. 25, no. 3, August, 1987)*

Show that

$$\sum_{i=0}^{k-1} \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} = \binom{n}{r},\ k \geq 1,\ 0 \leq r \leq k - 1 \leq n,$$

where the inner summation is over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n - i$ and $n_1 + \cdots + n_k = n - r$.

Solution by W. Moser, McGill University, Montreal, Canada

The number of solutions $(x_1, x_2, \ldots, x_{n-r}, i)$ of

(1) $\quad x_1 + x_2 + \cdots + x_{n-r} + i = r$

(where $x_1, x_2, \ldots, x_{n-r}, i$ are nonnegative integers)—or, equivalently, the number of ways of distributing r like objects into $n - r - 1$ unlike boxes—is $\binom{n}{r}$. This is well known and easy to prove. Let

(2) $\quad n_q = \#\{j \mid x_j = q - 1,\ j = 1, 2, \ldots, n - r\},\ q = 1, 2, \ldots, k,$

i.e., n_q is the number of x_j's in (1) equal to $q - 1$. Since

$$x_1, x_2, \ldots, x_{n-r}, \; i \in \{0, 1, \ldots, r\} \text{ and } k - 1 \geq r,$$

every x_j is counted once in the sum

(3) $\quad n_1 + n_2 + \cdots + n_k = n - r,$

while $x_1 + x_2 + \cdots + x_{n-r}$ is equal to

(4) $\quad n_2 + 2n_3 + \cdots + (k - 1)n_k = r - i.$

[Note that (3) and (4) are together equivalent to (3) and $n_1 + 2n_2 + \cdots + kn_k = n - i$.] Thus, every solution of (1) yields a solution of (3) satisfying (4). For each i ($i = 0, 1, \ldots, r$), how many solutions of (1) yield the same solution of (3)? Corresponding to a solution of (3) satisfying (4) there are

$$\binom{n - r}{n_1, n_2, \ldots, n_k}$$

linear displays of $n - r$ integers--n_1 0's, n_2 1's, ..., n_k $k - 1$'s--and these integers named from left to right $x_1, x_2, \ldots, x_{n-r}$ have sum $r - i$. The identity follows.

Also solved by P. Bruckman, G. Dinside, and the proposers & D. Antzoulakos.

Generally True!

H-413 *Proposed by Gregory Wulczyn, Bucknell U. (retired), Lewisburg, PA (Vol. 25, no. 3, August 1987)*

Let m, n be integers. If m and n have the same parity, show that

(1) $\quad (2m + 1)F_{2n+1} - (2n + 1)F_{2m+1} \equiv 0 \pmod 5$;

(2) $\quad (2m + 1)F_{2n+1} - (2n + 1)F_{2m+1} \equiv 0 \pmod{25}$ if either

(a) $2m + 1$ or $2n + 1$ is a multiple of 5, or
(b) $m \equiv n \equiv 0$ or $m \equiv n \equiv -1 \pmod 5$.

If m and n have the opposite parity, show that

(3) $\quad (2m + 1)F_{2n+1} + (2n + 1)F_{2m+1} \equiv 0 \pmod 5$;

(4) $\quad (2m + 1)F_{2n+1} + (2n + 1)F_{2m+1} \equiv 0 \pmod{25}$ if either

(a) $2m + 1$ or $2n + 1$ is a multiple of 5, or
(b) $m \equiv n \equiv 0$ or $m \equiv n \equiv -1 \pmod 5$.

Solution by Paul S. Bruckman, (formerly) Fair Oaks, CA

The indicated results are true, but under more general conditions. We prove the more general result. We define $D(m, n)$ for all integers m and n as follows:

(1) $\quad D(m, n) = (2m + 1)F_{2n+1} - (-1)^{m+n}(2n + 1)F_{2m+1}.$

Also, for all integers k, we define θ_k as follows:

(2) $\quad \theta_k = \dfrac{(-1)^k F_{2k+1}}{2k + 1}.$

Note:
(3) $\quad D(m, n) = (-1)^n(2m + 1)(2n + 1)(\theta_n - \theta_m).$

We now investigate the values of $\theta_k \pmod{25}$. Clearly, if $k \equiv 2 \pmod 5$, then $2k + 1 \equiv 0 \pmod 5$, so $\theta_k \pmod{25}$ and $\theta_k \pmod 5$ are not defined in this case. We find that $\theta_k \pmod{25}$ (as defined) is periodic, with period 50, and we may form the following table (mod 25), omitting values of k with $k \equiv 2 \pmod 5$:

k	$(2k+1)^{-1}$	(-1)	F_{2k+1}	θ_k	k	$(2k+1)^{-1}$	(-1)	F_{2k+1}	θ_k
1	17	-1	2	16	28	18	1	12	16
3	18	-1	13	16	29	14	-1	16	1
4	14	1	9	1	30	16	1	11	1
5	16	-1	14	1	31	2	-1	17	16
6	2	1	8	16	33	3	-1	3	16
8	3	1	22	16	34	4	1	19	1
9	4	-1	6	1	35	6	-1	4	1
10	6	1	21	1	36	12	1	18	16
11	12	-1	7	16	38	13	1	7	16
13	13	-1	18	16	39	19	-1	21	1
14	19	1	4	1	40	21	1	6	1
15	21	-1	19	1	41	22	-1	22	16
16	22	1	3	16	43	23	-1	8	16
18	23	1	17	16	44	9	1	14	1
19	9	-1	11	1	45	11	-1	9	1
20	11	1	16	1	46	7	1	13	16
21	7	-1	12	16	48	8	1	2	16
23	8	-1	23	16	49	24	-1	1	1
24	24	1	24	1	50	1	1	1	1
25	1	-1	24	1	51	17	-1	2	16
26	17	1	23	16	etc.				

Inspection of the foregoing table yields the following result:

(4) $\theta_k \equiv 1 \pmod{25}$ iff $k \equiv 0$ or $4 \pmod 5$;

$\theta_k \equiv 16 \pmod{25}$ iff $k \equiv 1$ or $3 \pmod 5$.

It follows from (3) that $D(m, n) \equiv 0 \pmod{25}$ if any of the following conditions on m and n (mod 5) hold:

$$(m, n) = (0, 0), (0, 4), (4, 0), (4, 4),$$
$$(1, 1), (1, 3), (3, 1), \text{ or } (3, 3).$$

This proves parts (2)(b) and (4)(b) of the problem, but gives more general conditions for which $D(m, n) \equiv 0 \pmod{25}$.

Now, if $m \equiv 2 \pmod 5$, then $2m + 1 \equiv 0 \pmod 5$ and $F_{2m+1} = 0 \pmod 5$. Letting $U_n = F_{2n+1} - (-1)^n(2n + 1)$ and $V_n = F_{2n+1} + (-1)^n(2n + 1)$, we may form the following table (mod 25), which is periodic with period 50:

m	$2m+1$	F_{2m+1}	$D(m, n)$
2	5	5	$5U_n$
7	-10	10	$-10U_n$
12	0	0	0
17	10	-10	$10U_n$
22	-5	-5	$-5U_n$
27	5	5	$5V_n$
32	-10	10	$-10V_n$
37	0	0	0
42	10	-10	$10V_n$
47	-5	-5	$-5V_n$
52	5	5	$5U_n$

From the table, we see that if $m \equiv 2 \pmod 5$, then $D(m, n) \equiv 0 \pmod{25}$ for all n only if $5 | U_n$ or $5 | V_n$, i.e., $F_{2n+1} \equiv \pm(2n + 1) \pmod 5$ for all n. To test this, we prepare the following table (mod 5), which has period 20:

k	F_k	$F_k + k$ or $F_k - k$*
1	1	$1 - 1 \equiv 0$
3	2	$3 + 2 \equiv 0$
5	0	$0 + 0 \equiv 0$
7	3	$7 + 3 \equiv 0$
9	4	$9 - 4 \equiv 0$
11	4	$11 + 4 \equiv 0$
13	3	$13 - 3 \equiv 0$
15	0	$15 + 0 \equiv 0$
17	2	$17 - 2 \equiv 0$
19	1	$19 + 1 \equiv 0$
21	1	$21 - 1 \equiv 0$

*whichever is applicable

Thus, $5 | U_n$ or $5 | V_n$ for all n, which proves that $D(m, n) \equiv 0 \pmod{25}$ if $m \equiv 2 \pmod 5$. Similarly, $D(m, n) \equiv 0 \pmod{25}$ if $n \equiv 2 \pmod 5$. This proves parts (2)(a) and (4)(a) of the problem. Thus, if $m \equiv 2$ or $n \equiv 2 \pmod 5$, $D(m, n) \equiv 0 \pmod 5$. On the other hand, if $m \not\equiv 2$ and $n \not\equiv 2 \pmod 5$, then $\theta_m \equiv \theta_n \equiv 1 \pmod 5$ (from the first table); in the latter case, therefore, $D(m, n) \equiv 0 \pmod 5$ also [using (3)]. This proves parts (1) and (3).

Also solved by L. Kuipers, L. Sohmer, and the proposer.

*****