ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

A. P. Hillman

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-652 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $\alpha = (1 + \sqrt{5})/2$,

$$S_1(n) = \sum_{k=1}^{n} \alpha^k \quad \text{and} \quad S_2(n) = \sum_{k=1}^{n} \alpha^{-k}.$$

Determine m as a function of n such that $\dfrac{S_1(n)}{S_2(n)} - \alpha F_m$ is a Fibonacci number.

B-653 *Proposed by Herta T. Freitag, Roanoke, VA*

The sides of a triangle are $a = F_{2n+3}$, $b = F_{n+3}F_n$, and $c = F_3F_{n+2}F_{n+1}$, with n a positive integer.

(i) Is the triangle acute, right, or obtuse?
(ii) Express the area as a product of Fibonacci numbers.

B-654 *Proposed by Alejandro Necochea, Pan American U., Edinburgh, TX*

Sum the infinite series

$$\sum_{k=1}^{\infty} \frac{1 + 2^k}{2^{2k}} F_k.$$

B-655 *Proposed by L. Kuipers, Sierre, Switzerland*

Prove that the ratio of integers x/y such that

$$\frac{F_{2n}}{F_{2n+2}} < \frac{x}{y} < \frac{F_{2n+1}}{F_{2n+3}}$$

and with smallest denominator y is $(F_{2n} + F_{2n+1})/(F_{2n+2} + F_{2n+3})$.

B-656 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Find a closed form for the sum

$$S_n = \sum_{k=0}^{n} w_k p^{n-k},$$

where w_n satisfies $w_n = pw_{n-1} - qw_{n-2}$ for n in $\{2, 3, \ldots\}$, with p and q nonzero constants.

B-657 *Proposed by Clark Kimberling, U. of Evansville, Evansville, IN*

Let m be an integer and $m \geq 3$. Prove that no two of the integers

$$k(mF_n + F_{n-1}) \text{ for } k = 1, 2, \ldots, m-1 \text{ and } n = 0, 1, 2, \ldots$$

are equal. Here $F_{-1} = 1$.

SOLUTIONS

Average Age of Fibonacci's Rabbits

B-628 *Proposed by David Singmaster, Polytechnic of the South Bank, London, England*

What is the present average age of Fibonacci's rabbits? (Recall that he introduced a pair of mature rabbits at the beginning of his year and that rabbits mature in their second month. Further, no rabbits died. Let us say that he did this at the beginning of 1202 and that he introduced a pair of 1-month-old rabbits. At the end of the first month, this pair would have matured and produced a new pair, giving us a pair of 2-month-old rabbits and a pair of 0-month-old rabbits. At the end of the second month we have a pair of 3-month-old rabbits and pairs of 1-month-old and of 0-month-old rabbits.) Before solving the problem, make a guess at the answer.

Solution by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

The solution to this problem is rather amazing. If n is the number of months within the interval (1^{st} Jan. 1202-1^{st} Nov. 1988), then the number of pairs of Fibonacci rabbits in the enclosure is currently F_{n+1}. On the basis of the growth rule, their average age A_n (in months) is

$$(1) \qquad A_n = \left(n + \sum_{i=1}^{n-2} iF_{n-1-i}\right) F_{n+1}.$$

By using the identity

$$(2) \qquad \sum_{i=1}^{N} iF_{k-i} = F_{k+3} - (N+2)F_{k-N+1} - F_{k-N},$$

the proof of which is omitted in this context, we can evaluate (1)

(3) $\quad A_n = (n + F_{n+2} - nF_2 - F_1)/F_{n+1} = (F_{n+2} - 1)/F_{n+1}.$

Since n is sufficiently large (> 9000), we have

$$A_n \approx \alpha = (1 + \sqrt{5})/2 \text{ months.}$$

Also solved by Charles Ashbacher, Paul Bruckman, John Cannell, Carl Libis, and the proposer.

Always at Least One Solution

B-629 *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

For which integers a, b, and c is it possible to find integers x and y satisfying

$$(x + y)^2 - cx^2 + 2(b - a + ac)x - 2(a - b)y + (a - b)^2 - ca^2 = 0?$$

Solution by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA

We prove more than what is asked; namely, for every possible choice of the ordered triple (a, b, c), we find the corresponding solution (x, y) satisfying the given equation.

The given equation can be written as:

$$(x + y)^2 - 2(a - b)(x + y) + (a - b)^2 = c(x - a)^2$$

or
$$[(x + y) - (a - b)]^2 = c(x - a)^2$$

or
$$[(x - a) + (y + b)]^2 = c(x - a)^2$$

The following cases are possible:

(i) If $c = 0$, then (x, y) takes infinitely many integral values, namely,

$$(x,\ a - b - x) \text{ where } a, b, x \text{ are arbitrary integers.}$$

Thus, with a, b as arbitrary integers and $c = 0$ the corresponding solution is

$$(x,\ a - b - x) \text{ where } x \text{ is any integer.}$$

(ii) If $x = a$, then $y = -b$ and c can be any arbitrary integer. Hence, with any choice of integral values of a, b and arbitrary integer c, we have $(a, -b)$ as the solution for (x, y).

(111) If $x \neq a$, $c \neq 0$, then it follows that c must be the square of an integer and $(x - a)$ must divide $(y + b)$. Consequently, if $c = n^2$ where n is a positive integer, then with a, b as arbitrary integers and $c = n^2$, we get two possible integral solutions:

$$[x,\ (n - 1)(x - a) - b] \quad \text{and} \quad [x,\ -(n + 1)(x - a) - b]$$

for (x, y) where x is an arbitrary integer.

Also solved by Paul Bruckman, L. Kuipers, Amitabha Tripathi, and the proposer.

Golden Geometric Progression

B-630 *Proposed by Herta T. Freitag, Roanoke, VA*

Let a and b be constants and define the sequences

$$\{A_n\}_{n=1}^{\infty} \quad \text{and} \quad \{B_n\}_{n=1}^{\infty}$$

by $A_1 = a$, $A_2 = b$, $B_1 = 2b - a$, $B_2 = 2a + b$, and $A_n = A_{n-1} + A_{n-2}$ and $B_n = B_{n-1} + B_{n-2}$ for $n \ge 3$.

(i) Determine a and b so that $(A_n + B_n)/2 = [(1 + \sqrt{5})/2]^n$.

(ii) For these a and b, obtain $(B_n + A_n)/(B_n - A_n)$.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

(i) Since $C_n = (A_n + B_n)/2$ satisfies the second-order recurrence relation

$$C_n = C_{n-1} + C_{n-2} \text{ for } n \ge 3,$$

C_1 and C_2 determine the sequence $\{C_n\}_{n=1}^{\infty}$. Solving the system of equations

$$b = C_1 = (1 + \sqrt{5})/2$$
$$a + b = C_2 = [(1 + \sqrt{5})/2]^2 = (3 + \sqrt{5})/2$$

we obtain $a = 1$ and $b = (1 + \sqrt{5})/2$.

(ii) For these a and b, we have

$$B_1 = \sqrt{5} = \sqrt{5}A_1 \quad \text{and} \quad B_2 = (5 + \sqrt{5})/2 = \sqrt{5}A_2.$$

So

$$B_n = \sqrt{5}A_n \quad \text{and} \quad (B_n - A_n)/2 = [(\sqrt{5} - 1)/2]A_n \text{ for all } n \ge 1.$$

Thus,

$$(B_n + A_n)/2 = [(\sqrt{5} + 1)/2]A_n$$

and

$$\frac{B_n + A_n}{B_n - A_n} = \frac{\sqrt{5} + 1}{\sqrt{5} - 1} = \left(\frac{\sqrt{5} + 1}{2}\right)^2.$$

Also solved by Charles Ashbacher, Paul Bruckman, Russell Euler, Piero Filipponi, L. Kuipers, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Amitabha Tripathi, and the proposer.

Closed Form

B-631 *Proposed by L. Kuipers, Sierre, Switzerland*

For N in $\{1, 2, \ldots\}$ and $N \ge m + 1$, obtain, in closed form,

$$u_N = \sum_{k=m+1}^{m+N} k(k-1)\cdots(k-m)\binom{n+k}{k}.$$

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

$$u_N = \frac{1}{n!}\sum_{k=m+1}^{m+N}\frac{(n+k)!}{(k-m-1)!} = \frac{(n+m+1)!}{n!}\sum_{k=m+1}^{m+N}\binom{n+k}{n+m+1}$$
$$= \frac{(n+m+1)!}{n!}\binom{n+m+N+1}{n+m+2}$$

Also solved by Paul Bruckman, Odoardo Brugia & Piero Filipponi, and the proposer.

Golden Determinant

B-632 *Proposed by H.-J. Seiffert, Berlin, Germany*

Find the determinant of the n by n matrix (x_{ij}) with $x_{ij} = (1 + \sqrt{5})/2$ for $j > i$, $x_{ij} = (1 - \sqrt{5})/2$ for $j < i$, and $x_{ij} = 1$ for $j = i$.

Solution by Hans Kappus, Rodersdorf, Switzerland

More generally, let us determine the characteristic polynomial

$$f_n(t) = \det(x_{ij} - t\delta_{ij}).$$

Subtracting the $(n - i)^{\text{th}}$ line from the $(n - i + 1)^{\text{th}}$ line $(i = 1, \ldots, n - 1)$ we obtain the determinant

$$f_n(t) = \begin{vmatrix} 1 - t & \alpha & \ldots \alpha & \alpha \\ \beta - 1 + t & 1 - \alpha - t & \ldots 0 & 0 \\ 0 & \beta - 1 + t & \ldots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots \beta - 1 + t & 1 - \alpha - t \end{vmatrix}$$

which, after expanding with respect to the n^{th} column, may be written as

$$\begin{aligned} f_n(t) &= \alpha(1 - \beta - t)^{n-1} + (1 - \alpha - t)f_{n-1}(t) \\ &= \alpha(\alpha - t)^{n-1} + (\beta - t)f_{n-1}(t). \end{aligned}$$

Because of symmetry, we may interchange α and β and eliminate $f_{n-1}(t)$. Thus, we arrive at

$$\begin{aligned} f_n(t) &= (1/\sqrt{5})\{\alpha(\alpha - t)^n - \beta(\beta - t)^n\} \\ &= \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\frac{\alpha^{k+1} - \beta^{k+1}}{\sqrt{5}} t^{n-k} = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}F_{k+1}t^{n-k}. \end{aligned}$$

Therefore, the solution to the original problem is given by $f_n(0) = F_{n+1}$.

Editor's Note: Bob Prielipp pointed out that B-632 is a special case of the determinant of Problem A-2 of the 1978 W. L. Putnam Mathematical Competition. (The solution is in *American Mathematical Monthly*, Nov. 1979, p. 753.)

Also solved by Paul Bruckman, Odoardo Brugia & Piero Filipponi, Russell Euler, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Bob Prielipp, Sahib Singh, Amitabha Tripathi, and the proposer.

Ratio of Series

B-633 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $n \geq 2$ be an integer and define

$$A_n = \sum_{k=0}^{\infty} \frac{F_k}{n^k}, \quad B_n = \sum_{k=0}^{\infty} \frac{L_k}{n^k}.$$

Prove that $B_n/A_n = 2n - 1$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

$$F_k = (\alpha^k - \beta^k)/\sqrt{5}$$

and

$$L_k = \alpha^k + \beta^k \text{ where } \alpha = (1 + \sqrt{5})/2 \text{ and } \beta = (1 - \sqrt{5})/2.$$

The infinite geometric series

$$C_n = \sum_{k=0}^{\infty} \left(\frac{\alpha}{n}\right)^k \quad \text{and} \quad D_n = \sum_{k=0}^{\infty} \left(\frac{\beta}{n}\right)^k$$

both converge since the absolute value of each of their common ratios is less than 1. (Notice that the condition $n \geq 2$ is needed to insure the convergence of the first series.) Thus,

$$(1) \qquad A_n = \frac{1}{\sqrt{5}}(C_n - D_n) = \frac{1}{\sqrt{5}}\left(\frac{n}{n - \alpha} - \frac{n}{n - \beta}\right) = \frac{n}{\sqrt{5}} \frac{\alpha - \beta}{(n - \alpha)(n - \beta)}$$

$$= \frac{n}{(n - \alpha)(n - \beta)} \quad \text{since } \alpha - \beta = \sqrt{5}$$

and

$$(2) \qquad B_n = C_n + D_n = \frac{n}{n - \alpha} + \frac{n}{n - \beta} = \frac{n(2n - (\alpha + \beta))}{(n - \alpha)(n - \beta)}$$

$$= \frac{n(2n - 1)}{(n - \alpha)(n - \beta)} \quad \text{because } \alpha + \beta = 1.$$

Therefore, $B_n/A_n = 2n - 1$.

Also solved by Paul Bruckman, Russell Euler, Herta T. Freitag, Jay Hendel, Hans Kappus, L. Kuipers, Carl Libis, H.-J. Seiffert, Sahib Singh, Amitabha Tripathi, and the proposer.

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