# ODD NONUNITARY PERFECT NUMBERS

Peter Hagis, Jr.
Temple University, Philadelphia, PA 19122
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## 1. Introduction

Throughout this paper lower-case letters will be used to denote natural numbers, with $p$ and $q$ always representing primes. As usual, $(c, d)$ will symbolize the greatest common divisor of $c$ and $d$. If $cd = n$ and $(c, d) = 1$, then $d$ is said to be a unitary divisor of $n$ and we write $d \| n$. $\sigma(n)$ and $\sigma^*(n)$ denote, respectively, the sum of the divisors and unitary divisors of $n$. Both $\sigma$ and $\sigma^*$ are multiplicative, and $\sigma(p^e) = 1 + p + \cdots + p^e$ while $\sigma^*(p^e) = 1 + p^e$.

In [1] Ligh & Wall have defined $d$ to be a nonunitary divisor of $n$ if $cd = n$ and $(c, d) > 1$. If $\sigma^\#(n)$ denotes the sum of the nonunitary divisors of $n$, it is immediate that $\sigma^\#(n) = \sigma(n) - \sigma^*(n)$. It is easy to see that $\sigma^\#$ is not multiplicative, and that $\sigma^\#(n) = 0$ if and only if $n$ is squarefree. Now, $n$ has a unique representation of the form $n = \overline{n} \cdot n^\#$ where $(\overline{n}, n^\#) = 1$, $\overline{n}$ is squarefree, and $n^\#$ is powerful. (The value of $\overline{n}$ is 1 if $n$ is powerful, $n^\# = 1$ if $n$ is squarefree, and $1 = 1 \cdot 1$.) It follows easily that

$$\sigma^\#(n) = \sigma(\overline{n}) \cdot \sigma^\#(n^\#)$$

so that

$$\sigma^\#(n) = \prod_{p \| n} (1 + p)\left\{\prod_{p^e \| n} (1 + p + \cdots + p^e) - \prod_{p^e \| n} (1 + p^e)\right\} \tag{1}$$

where $e > 1$.

Ligh & Wall [1] say that $n$ is a $k$-fold nonunitary perfect number if $\sigma^\#(n) = kn$. In particular, if $\sigma^\#(n) = n$, then $n$ is said to be a nonunitary perfect number. The integers $m$ and $n$ are nonunitary amicable numbers if $\sigma^\#(m) = n$ and $\sigma^\#(n) = m$. All known $k$-fold nonunitary perfect numbers and all known nonunitary amicable pairs are even. In the present paper we initiate the study of *odd* nonunitary perfect numbers. Nonunitary aliquot sequences will also be discussed.

## 2. Odd Nonunitary Perfect Numbers

We begin this section by proving the following

*Theorem 1:* The value of $\sigma^\#(n)$ is odd if and only if $n = 2^\alpha M^2$ where $(M, 2) = 1$, $M > 1$, $\alpha \geq 0$.

*Proof:* Suppose that $\sigma^\#(n)$ is odd and $n = 2^\alpha K$ where $(K, 2) = 1$ and $\alpha \geq 0$. Then $K \geq 3$ since $\sigma^\#(2^0) = \sigma^\#(2) = 0$ and $\sigma^\#(2^\alpha)$ is even if $\alpha \geq 2$. Since $2 \mid (1 + p^e)$ if $p$ is odd, and since $2 \mid (1 + p + \cdots + p^e)$ if and only if $e$ is odd, it follows easily from (1) [since $\sigma^\#(n)$ is odd] that $K = M^2$ *and* $n = 2^\alpha M^2$. Now suppose that $n = 2^\alpha M^2$ where $(M, 2) = 1$, $M > 1$, $\alpha \geq 0$. Since $(1 + p^e)$ is even and $(1 + p + \cdots + p^e)$ is odd if $e$ is even and $p$ is odd, it follows from (1) that $\sigma^\#(n) = \sigma^\#(2^\alpha M^2)$ is odd for $\alpha \geq 0$.

The following corollaries are immediate consequences of Theorem 1.

*Corollary 1:* If $n$ is an odd nonunitary perfect number (or an odd $k$-fold nonunitary perfect number where $k$ is odd), then $n = M^2$.

*Corollary 2:* If $m$ and $n$ are nonunitary odd amicable numbers, then $m = M^2$ and $n = N^2$.

*Corollary 3:* If $m$ and $n$ are nonunitary amicable numbers of opposite parity ($2|m$ and $2\nmid n$), then $m = 2^{\alpha}M^2$ where $(M,2) = 1$, $\alpha \geq 1$.

Now suppose that $n$ is an odd nonunitary perfect number. From Corollary 1, $n = p_1^{e_1}p_2^{e_2} \ldots p_t^{e_t}$ where $2|e_i$ for $i = 1, 2, \ldots, t$. From (1), we have

$$n = \prod_{i=1}^{t}(1 + p_i + \cdots + p_i^{e_i}) - \prod_{i=1}^{t}(1 + p_i^{e_i}), \tag{2}$$

and it follows that

$$1 = \prod_{i=1}^{t}(1 + p_i^{-1} + \cdots + p_i^{-e_i}) - \prod_{i=1}^{t}(1 + p_i^{-e_i}). \tag{3}$$

Therefore,

$$1 < \prod_{i=1}^{t}(1 + p_i^{-1} + p_i^{-2} + \cdots) - \prod_{i=1}^{t} 1$$

or

$$\prod_{p|n} p/(p - 1) > 2. \tag{4}$$

It is well known that (4) holds for (ordinary) odd perfect numbers. Let $\omega(n)$ denote the number of distinct prime factors of $n$. From the table given by Norton in [2], we have

*Proposition 1:* Suppose that $n$ is a nonunitary odd perfect number. Then $\omega(n) \geq 3$. If $3\nmid n$, then $\omega(n) \geq 7$ and

$$n \geq (5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23)^2 > 10^{15}.$$

If $(15,n) = 1$, then $\omega(n) \geq 15$ and

$$n \geq (7 \cdot 11 \cdot 13 \cdot \cdots \cdot 59 \cdot 61)^2 > 10^{43}.$$

A computer search was made for all odd nonunitary perfect numbers less than $10^{15}$. None was found. Therefore, we have

*Proposition 2:* If $n$ is an odd nonunitary perfect number, then $n > 10^{15}$.

If $n = p_1^{e_1}p_2^{e_2} \ldots p_t^{e_t}$ (where $2|e_i$) is an odd nonunitary perfect number and $1 \leq f_i \leq e_i$, then it follows easily from (3) that

$$\prod_{i=1}^{t}(1 + p_i^{-1} + \cdots + p_i^{-f_i}) - \prod_{i=1}^{t}(1 + p_i^{-f_i}) \leq 1. \tag{5}$$

In particular, if $n$ is an odd nonunitary perfect number,

$$\prod_{p|n}(1 + p^{-1} + p^{-2}) - \prod_{p|n}(1 + p^{-2}) \leq 1. \tag{6}$$

*Lemma 1:* Suppose that $N = p_1^{a_1} \ldots p_r^{a_r}q_1^{b_1} \ldots q_s^{b_s} = RS$ where $(R,S) = 1$ and $S \geq 1$. If

(7) $$\sigma^{\#}(p_1^{c_1} \cdots p_r^{c_r})/(p_1^{c_1} \cdots p_r^{c_r})$$
$$= \prod_{i=1}^{r}(1 + p_i^{-1} + \cdots + p_i^{-c_i}) - \prod_{i=1}^{r}(1 + p_i^{-c_i}) > 1$$

where $2 \le c_i \le a_i$ for $i = 1, 2, \ldots, r$, then $N$ is not an odd nonunitary perfect number.

*Proof:* If $W - V > 1$ and $V > 0$, it is easy to see that

$$W(1 + p^{-1} + \cdots + p^{-b}) - V(1 + p^{-b}) > W - V > 1.$$

It follows from (7) that $N$ cannot satisfy the inequality (5). Therefore, $N$ is not an odd nonunitary perfect number.

Now suppose that $n$ is an odd nonunitary perfect number and $3|n$. Then

$$n = 3^{e_1}p_2^{e_2} \cdots p_t^{e_t} \text{ where } 2|e_i.$$

Since $1 + 3 + \cdots + 3^{e_1} \equiv 1 + 3^{e_1} \equiv 1 \pmod 3$ and since $1 + p^e \equiv -1 \pmod 3$ if $p > 3$ and $2|e$, it follows from (1) that

(8) $$\sigma^{\#}(n) = n \equiv 0 \equiv \prod_{p^e \| n}(1 + p + \cdots + p^e) + (-1)^t \pmod 3 \text{ where } p > 3.$$

If $p \equiv -1 \pmod 3$, then $1 + p + \cdots + p^e \equiv 1 \pmod 3$ if $e$ is even; if $p \equiv 1 \pmod 3$, then $1 + p + \cdots + p^e \equiv 0, -1, 1 \pmod 3$ according as $e \equiv 2, 4, 6 \pmod 6$, respectively. The following lemma is an immediate consequence of (8) and the preceding remark.

*Lemma 2:* Suppose that $n$ is an odd nonunitary perfect number such that $3|n$ and $\omega(n) = t$. If $p^e\|n$ and $p \equiv 1 \pmod 3$, then $e \ge 4$. [More precisely, $e \equiv 0, 4 \pmod 6$.] If $2|t$, then $n$ has an odd number of components $p^e$ such that $p \equiv 1 \pmod 3$ and $e \equiv 4 \pmod 6$. If $2\nmid t$, then $n$ has an even number of components $p^e$ such that $p \equiv 1 \pmod 3$ and $e \equiv 4 \pmod 6$.

Now assume that $n$ is an odd nonunitary perfect number such that $3 \cdot 5 \cdot 7|n$. From Lemma 2, $7^4|n$. Suppose that $3^4|n$. Then, since $\sigma^{\#}(3^4 5^2 7^4)/3^4 5^2 7^4 > 1$, Lemma 1 yields a contradiction. Therefore, $3^2\|n$. Since $\sigma^{\#}(3^2 5^2 7^4 13^2)/3^2 5^2 7^4 13^2 > 1$, Lemma 1 shows that $13\nmid n$; and since $1 + 3 + 3^2 = 13$ and $1 + 5^2 = 2 \cdot 13$, we conclude from (2) that $5^2\not\| n$ so that $5^4|n$.

If $p > 7$, let $F(p) = \sigma^{\#}(3^2 5^4 7^4 p^2)/3^2 5^4 7^4 p^2$. It is easy to verify that $F$ is a monotonic decreasing function of $p$ and that $F(271) > 1$. [($F(277) < 1$.] We have proved

*Proposition 3:* If $n$ is an odd nonunitary perfect number and if $3 \cdot 5 \cdot 7|n$, then $3^2\|n$ and $5^4 7^4|n$. Also, $p\nmid n$ if $11 \le p \le 271$.

*Theorem 2:* If $n$ is an odd nonunitary perfect number, then $\omega(n) \ge 4$.

*Proof:* Assume that $\omega(n) < 4$. Then from Proposition 1, $\omega(n) = 3$ and $3|n$. Since $(3/2)(7/6)(11/10) < 2$ and $x/(x - 1)$ is monotonic decreasing for $x > 1$, it follows from (4) that $5|n$. Since $(3/2)(5/4)(17/16) < 2$, $p\nmid n$ if $p \ge 17$.

Assume that $3 \cdot 5 \cdot 7|n$. From Lemma 2, $3^2\|n$ and it follows easily from (3) that $1 < (13/9)(5/4)(7/6) - (10/9)$. This is a contradiction.

Now suppose that $3 \cdot 5 \cdot 13|n$. If $3^2\|n$, then, from (3),

$$1 < (13/9)(5/4)(13/12) - (10/9).$$

If $5^2 \| n$, then

$$1 < (3/2)(31/25)(13/12) - (26/25).$$

In each case, we have a contradiction. Therefore, $3^4 5^4 13^2 | n$. But,

$$\sigma^{\#}(3^4 5^4 13^2)/3^4 5^4 13^2 > 1$$

and, from Lemma 1, $n$ is not a nonunitary perfect number.

Finally, assume that $3 \cdot 5 \cdot 11 | n$. If $3^2 \| n$, then, from (3),

$$1 < (13/9)(5/4)(11/10) - (10/9)$$

and we have a contradiction. If $3^4 \| n$, then, since $1 + 3 + 3^2 + 3^3 + 3^4 = 11^2$ and $1 + 3^4 = 82$, it follows from (2) that $0 \equiv -5(1 + 5^e) \pmod{11}$. This is impossible since $11 \nmid (1 + 5^e)$ if $2 | e$. Therefore, $3^6 | n$. Now assume that $5^2 \| n$. If $11^4 | n$, then, since

$$\sigma^{\#}(3^6 5^2 11^4)/3^6 5^2 11^4 > 1,$$

we have a contradiction. Therefore, $11^2 \| n$. Since $n = 3^e 5^2 11^2$, it follows from (2) that

$$5^2 \cdot 11^2 \cdot 3^e = 31 \cdot 133 \cdot (3^{e+1} - 1)/2 - 26 \cdot 122 \cdot (1 + 3^e).$$

Therefore, $25 \cdot 3^e = 10467$ and we have a contradiction. We conclude that $5^4 | n$. But

$$\sigma^{\#}(3^6 5^4 11^2)/3^6 5^4 11^2 > 1$$

and, from Lemma 1, $n$ is not a nonunitary perfect number.

## 3. Nonunitary Aliquot Sequences

A $t$-tuple of distinct natural numbers $(n_0; n_1; \ldots; n_{t-1})$ with $n_i = \sigma^{\#}(n_{i-1})$ for $i = 1, 2, \ldots, t - 1$ and $n_0 = \sigma^{\#}(n_{t-1})$ is called a *nonunitary t-cycle*. A nonunitary 1-cycle is a nonunitary perfect number; a nonunitary 2-cycle is a nonunitary amicable pair. A search was made for all nonunitary $t$-cycles with $t > 2$ and $n_0 \le 10^6$. One was found:

(619368; 627264; 1393551)

The *nonunitary aliquot sequence* $\{n_i\}$ with leader $n$ is defined by

$$n_0 = n,\ n_1 = \sigma^{\#}(n_0),\ n_2 = \sigma^{\#}(n_1),\ \ldots,\ n_i = \sigma^{\#}(n_{i-1}),\ \ldots .$$

Such a sequence is said to be *terminating* if $n_k$ is squarefree for some index $k$ (so that $n_i = 0$ for $i > k$). [We define $\sigma^{\#}(0) = 0$.] A nonunitary aliquot sequence is said to be *periodic* if an index $k$ exists such that $(n_k; n_{k+1}; \ldots; n_{k+t-1})$ is a nonunitary $t$-cycle. A nonunitary aliquot sequence which is neither terminating nor periodic is *unbounded*. Whether or not unbounded nonunitary aliquot sequences exist is an open question.

An investigation was made of all nonunitary aliquot sequences with leader $n \le 10^6$. About 40 minutes of computer time was required. 740671 sequences were found to be terminating; 1440 were periodic (194 ended in 1-cycles, 1195 in 2-cycles, and 51 in 3-cycles); and in 257889 cases an $n_k > 10^{12}$ was encountered and (for practical reasons) the sequence was terminated with its final behavior undetermined. As was pointed out by the referee, since there are 607926 squarefree numbers between 1 and $10^6$, more than 82% of the 740671 terminating sequences were guaranteed to terminate *before* the investigation just described even began. From this perspective we see that the behavior of only about one-third of the "doubtful" sequences with leaders less than $10^6$ has been determined. The first sequence with unknown behavior has leader $n_0 = 792$.

$n_{52} = 1{,}780{,}270{,}202{,}880$ is the first term of this sequence which exceeds $10^{12}$.

## References


1. S. Ligh & C. R. Wall. "Functions of Nonunitary Divisors." *Fibonacci Quarterly* *25.4* (1987):333-338.
2. K. K. Norton. "Remarks on the Number of Factors of an Odd Perfect Number." *Acta Arithmetica* *6* (1961):365-374.


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*Announcement*

# FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

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The FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Wake Forest University, Winston-Salem, N.C., from July 30 to August 3, 1990. This Conference is sponsored jointly by the Fibonacci Association and Wake Forest University.

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