ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-440 *Proposed by T. V. Padmakumar, Trivandrum, India*

If $a_1, a_2, \ldots, a_m, n$ are positive integers such that $n > a_1, a_2, \ldots, a_m$ and $\emptyset(n) = m$ and a_i is relatively prime to n for $i = 1, 2, 3, \ldots, m$, prove

$$\left(\prod_{i=1}^{m} a_i\right)^2 \equiv 1 \pmod{n}.$$

H-441 *Proposed by Albert A. Mullin, Huntsville, AL*

By analogy with palindrome, a *validrome* is a sentence, formula, relation, or verse that remains valid whether read forward or backward. For example, relative to *prime* factorization, 341 is a factorably validromic number since $341 = 11 \cdot 31$, and when backward gives $13 \cdot 11 = 143$, which is also correct. (1) What is the largest factorably validromic square you can find? (2) What is the largest factorably validromic square, *avoiding* palindromic numbers, you can find? Here are three examples of factorably validromic squares:

$13 \cdot 13$, $101 \cdot 101$, $311 \cdot 311$.

H-442 *Proposed by Piero Filipponi, Rome, Italy*

Prove that the congruence

$$\prod_{i=1}^{(d-3)/2} (2i+1)^2 \equiv \begin{cases} 1 \pmod{d} \text{ if } (d+1)/2 \text{ is even} \\ -1 \pmod{d} \text{ if } (d+1)/2 \text{ is odd} \end{cases}$$

holds if and only if d is an odd prime.

SOLUTIONS

A Fifth

H-365 *Proposed by Larry Taylor, Rego Park, NY*
(Vol. 22, no. 1, February 1984)

Call a Fibonacci-Lucas identity divisible by 5 if every term of the identity is divisible by 5. Prove that, for every Fibonacci-Lucas identity not divisible by 5, there exists another Fibonacci-Lucas identity not divisible by 5 that can be derived from the original identity in the following way:

1) If necessary, restate the original identity in such a way that a derivation is possible.

2) Change one factor in every term of the original identity from F_n to L_n or from L_n to $5F_n$ in such a way that the result is also an identity. If the resulting identity is not divisible by 5, it is the derived identity.

3) If the resulting identity is divisible by 5, change one factor in every term of the original identity from L_n to F_n or from $5F_n$ to L_n in such a way that the result is also an identity. This is equivalent to dividing every term of the first resulting identity by 5. Then, the second resulting identity is the derived identity.

For example, $F_n L_n = F_{2n}$ can be restated as $F_n L_n = F_{2n} \pm F_0(-1)^n$. This is actually two distinct identities, of which the derived identities are

$$L_n^2 = L_{2n} + L_0(-1)^n \quad \text{and} \quad 5F_n^2 = L_{2n} - L_0(-1)^n.$$

Partial solution (Outline) by the proposer

Define a Fibonacci-Lucas equation as an algebraic equation in one unknown in which one of the roots is equal to $(1+\sqrt{5})/2$. Call a Fibonacci-Lucas equation divisible by $\sqrt{5}$ if every term of the equation is of the form $(5a + b\sqrt{5})/2$ where a and b are integers.

Define a Fibonacci-Lucas identity as the sum of a finite number of terms equated to zero, each of which terms is the product of a finite number of factors, one of which factors is either a Fibonacci or a Lucas number. Call a Fibonacci-Lucas identity divisible by 5 if every term of the identity is of the form $5a$ where a is an integer.

Theorem 1: There are only eight three-term Fibonacci-Lucas identities not divisible by 5.

Theorem 2: Every Fibonacci-Lucas identity can be derived from a three-term Fibonacci-Lucas identity by algebraic manipulation.

Theorem 3: From every Fibonacci-Lucas equation not divisible by $\sqrt{5}$ it is possible to derive two Fibonacci-Lucas identities not divisible by 5.

Theorem 4: There are only four three-term Fibonacci-Lucas equations not divisible by $\sqrt{5}$.

Theorem 5: Every Fibonacci-Lucas equation can be derived from a three-term Fibonacci-Lucas equation by algebraic manipulation.

Theorem 6: From every Fibonacci-Lucas identity not divisible by 5 it is possible to derive another Fibonacci-Lucas identity not divisible by 5 and a Fibonacci-Lucas equation not divisible by $\sqrt{5}$.

Comment: Theorem 6 uses Theorems 1 through 5 as lemmas; the proof of Theorem 6 is the complete solution of this problem.

Reference: L. Taylor. Partial solution of Problem H-365 (first segment). *Fibonacci Quarterly* 27.2 (1989):188-89.

Divide and Conquer

H-418 *Proposed by Lawrence Somer, Washington, D.C.*
(Vol. 26, no. 1, February, 1988)

Let $m > 1$ be a positive integer. Suppose that m itself is a general period of the Fibonacci sequence modulo m; that is $F_{n+m} \equiv F_n \pmod{m}$ for all nonnegative integers n. Show that $24 | m$.

Solution by Paul Bruckman, Edmonds, WA

Let a and b denote the usual Fibonacci constants; we deal with congruences in $F(\sqrt{5})$, modulo some integer, in the normal way. Given m as defined, we may suppose that

(1) $\quad a^m \equiv c, \quad b^m \equiv d \pmod{m}$.

Setting $n = 0$ in the original congruence, we have

(2) $\quad m | F_m$.

Thus, (1) and (2) imply that $c \equiv d \pmod{m}$. Also $a^{m+1} \equiv ca$, $b^{m+1} \equiv cb \pmod{m}$, so $F_{m+1} \equiv c \pmod{m}$. However, setting $n = 1$ in the original congruence, we have

(3) $\quad F_{m+1} \equiv 1 \pmod{m}$.

Therefore, $c = d = 1$, i.e.,

(4) $\quad a^m \equiv b^m \equiv 1 \pmod{m}$.

Now, a result of Jarden [1] states that

(5) $\quad m | F_m$, $m > 1$ implies either $5 | m$ or $12 | m$.

Note that $a = \frac{1}{2}(1 + \sqrt{5}) \equiv 2^{-1} \equiv 3 \pmod{5}$; also, $a^2 \equiv 4$, $a^3 \equiv 2$, and $a^4 \equiv 1 \pmod{5}$. Hence,

(6) $\quad a^r \equiv 1 \pmod{5}$ iff $4 | r$.

Thus, $a^r \equiv 1 \pmod{20}$ only if $4 | r$. But $a^4 = 2 + 3a = 2^{-1}(7 + 3\sqrt{5})$, and $a^8 = 13 + 21a = 2^{-1}(47 + 21\sqrt{5})$, neither of which expression is defined (mod 20); on the other hand, $a^{12} \equiv 89 + 144a = 2^{-1}(322 + 144\sqrt{5}) = 161 + 36\sqrt{20} \equiv 1 \pmod{20}$. Hence,

(7) $\quad a^r \equiv 1 \pmod{20}$ iff $12 | r$.

Suppose now that $5 | m$. Then $a^m \equiv 1 \pmod{m}$, by (4), so $a^m \equiv 1 \pmod{5}$, which implies $4 | m$, by (6); hence $20 | m$. Then $a^m \equiv 1 \pmod{20}$, so $12 | m$, by (7). Therefore, for m as defined,

(8) $\quad 5 | m$ implies $60 | m$.

Therefore, by Jarden's result in (5), we see that $3 | m$ in any event.

Next, we observe that

$$a^2 = 1 + a = 2^{-1}(3 + \sqrt{5}) \equiv 2\sqrt{5} \equiv -\sqrt{5} \pmod{3};$$
$$a^3 = 1 + 2a \equiv 1 - a = b \pmod{3}; \; a^4 \equiv ab \equiv -1 \pmod{3};$$
$$a^5 \equiv -a \pmod{3}; \; a^6 \equiv \sqrt{5} \pmod{3};$$
$$a^7 \equiv -b \pmod{3}; \; a^8 \equiv 1 \pmod{3}.$$

Therefore

(9) $\quad a^s \equiv 1 \pmod{3}$ iff $8 | s$.

Since $3 | m$, $a^m \equiv 1 \pmod{3}$, which implies $8 | m$, by (9); hence, $24 | m$. Q.E.D.

1. Dov Jarden. "Recurring Sequences." *Riveon Lematematika*, 3rd ed. (1973), Theorem F, p. 72.

Also solved by R. Jeannin, L. Kuipers, C. Long, P. Tzermias, and the proposer.

Pell-Mell

H-419 *Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 26, no. 1, February 1988)*

Let $P_0, P_1, \ldots$ be the sequence of Pell numbers defined by

$$P_0 = 0, \; P_1 = 1, \; P_n = 2P_{n-1} + P_{n-2} \text{ for } n \in \{2, 3, \ldots\}.$$

Show that

(a) $9\sum_{k=0}^{n} kF_kP_k = 3(n+1)(F_nP_{n+1} + F_{n+1}P_n) - F_{n+2}P_{n+2} - F_nP_n + 2,$

(b) $9\sum_{k=0}^{n} kL_kP_k = 3(n+1)(L_nP_{n+1} + L_{n+1}P_n) - L_{n+2}P_{n+2} - L_nP_n,$

(c) $F_{m+n+2}P_{n+2} + F_{m+n}P_n \equiv 3(n+1)F_m + L_m \pmod 9,$

(d) $L_{m+n+2}P_{n+2} + L_{m+n}P_n \equiv 3(n+1)L_m + 5F_m \pmod 9,$

where n is a nonnegative integer and m any integer.

Solution by the proposer

Remark: (c) and (d) contain interesting special cases.

1) Taking $m = -n$ and using $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^nL_n$ in (c) yields

$$P_{n+2} \equiv (-1)^{n+1}(3(n+1)F_n - L_n) \pmod 9.$$

2) Taking $m = -(n+1)$ and using $P_{n+2} - P_n = 2P_{n+1}$ in (d) yields

$$2P_{n+1} \equiv (-1)^{n+1}(3(n+1)L_{n+1} - 5F_{n+1}) \pmod 9$$

or, after replacing n by $n - 1$

$$2P_n \equiv (-1)^n(3nL_n - 5F_n) \pmod 9.$$

3) Taking $m = -(n+1)$ in (c) and then replacing n by $n - 1$ yields

$$P_{n+1} + P_{n-1} \equiv (-1)^{n+1}(3nF_n - L_n) \pmod 9.$$

4) Taking $m = -n$ in (d) yields

$$3P_{n+2} + 2P_n \equiv (-1)^n(3(n+1)L_n - 5F_n) \pmod 9.$$

Let (G_n) denote either the sequence of Fibonacci or Lucas numbers. Then

$$\begin{aligned}
G_{n+3}P_{n+3} &= (G_{n+2} + G_{n+1})(2P_{n+2} + P_{n+1}) \\
&= 2G_{n+2}P_{n+2} + G_{n+2}P_{n+1} + 2G_{n+1}P_{n+2} + G_{n+1}P_{n+1} \\
&= G_{n+2}P_{n+2} + G_{n+2}(P_{n+2} + P_{n+1}) + 2G_{n+1}P_{n+2} + G_{n+1}P_{n+1} \\
&= G_{n+2}P_{n+2} + G_{n+2}(3P_{n+1} + P_n) + 2G_{n+1}P_{n+2} + G_{n+1}P_{n+1} \\
&= G_{n+2}P_{n+2} + 3G_{n+2}P_{n+1} + G_{n+2}P_n + 2G_{n+1}P_{n+2} + G_{n+1}P_{n+1} \\
&= G_{n+2}P_{n+2} + 3(G_{n+2}P_{n+1} + G_{n+1}P_{n+2}) - G_{n+1}P_{n+1} + G_{n+2}P_n \\
&\qquad - G_{n+1}(P_{n+2} - 2P_{n+1}) \\
&= G_{n+2}P_{n+2} + 3(G_{n+2}P_{n+1} + G_{n+1}P_{n+2}) - G_{n+1}P_{n+1} + G_nP_n \\
&\qquad + G_{n+1}(P_n - P_{n+2} + 2P_{n+1})
\end{aligned}$$

which yields

$$(1)\qquad G_{n+2}P_{n+2} + G_nP_n + 3(G_{n+1}P_{n+2} + G_{n+2}P_{n+1}) = G_{n+3}P_{n+3} + G_{n+1}P_{n+1}.$$

Now we are able to prove (a) and (b) by induction on n.

Proof of (a) and (b): Obviously (a) and (b) hold for $n = 0$. To show that both hold for $n + 1$ if they hold for n, we have to prove the equation

$$(*)\qquad \begin{aligned}&3(n+1)(G_nP_{n+1} + G_{n+1}P_n) - G_{n+2}P_{n+2} - G_nP_n + 9(n+1)G_{n+1}P_{n+1} \\ &= 3(n+2)(G_{n+1}P_{n+2} + G_{n+2}P_{n+1}) - G_{n+3}P_{n+3} - G_{n+1}P_{n+1}.\end{aligned}$$

Using

$$\begin{aligned}G_nP_{n+1} + G_{n+1}P_n + 3G_{n+1}P_{n+1} &= G_nP_{n+1} + G_{n+1}P_n + 2G_{n+1}P_{n+1} + G_{n+1}P_{n+1} \\ &= (G_n + G_{n+1})P_{n+1} + G_{n+1}(2P_{n+1} + P_n) = G_{n+1}P_{n+2} + G_{n+2}P_{n+1}\end{aligned}$$

and (1), we get (*).

Proof of (c) and (d): In [1] it is proved that

(2) $$3\sum_{k=0}^{n} F_k P_k = F_n P_{n+1} + F_{n+1} P_n$$

and

(3) $$3\sum_{k=0}^{n} L_k P_k = L_n P_{n+1} + L_{n+1} P_n - 2,$$

which shows that 3 divides the right side of (2) and (3). Thus, from (a) and (b) we easily obtain

(4) $$F_{n+2} P_{n+2} + F_n P_n \equiv 2 \text{ (mod } 9,$$

(5) $$L_{n+2} P_{n+2} + L_n P_n \equiv 6(n+1) \pmod{9}.$$

Now, if m is any integer, then we multiply (4) by L_m, (5) by F_m, and add the obtained congruences by using the formula $F_k L_m + L_k F_m = 2F_{m+k}$. Then we divide the obtained congruence by 2 [note that GCD(2, 9) = 1] to get (c).

To obtain (d) we multiply (4) by $5F_m$, (5) by L_m and add the obtained congruences by using the formula $5F_k F_m + L_k L_m = 2L_{m+k}$. Now, we again divide the obtained congruence to get (d). This completes the solution.

1. P. S. Bruckman. Solution of B565-B566. *Fibonacci Quarterly* 25.1 (1987):87-88.

Also solved by P. Bruckman, C. Georghiou, R. Andre-Jeannin, L. Kuipers, and G. Wulczyn.

Two Two Much

H-420 *Proposed by Peter Kiss, Eger, Hungary, and Andreas N. Philippou, Patras, Greece (Vol. 26, no. 1, February 1988)*

Show that

(1) $$\sum_{n=1}^{\infty} \frac{2^{2^{n-1}}}{2^{2^n} - 1} = 1.$$

Solution (and Generalization) by H. M. Srivastava, Victoria, Canada

It can easily be seen, by mathematical induction, that (see [1], Example 15, p. 24)

(2) $$\sum_{n=1}^{N} \frac{x^{2^{n-1}}}{x^{2^n} - 1} = \frac{1}{x-1} - \frac{1}{x^{2^N} - 1} \quad (x \neq 1).$$

Now let $N \to \infty$ in cases when $|x| > 1$ and $|x| < 1$, separately, and (2) leads us immediately to the sum

(3) $$\sum_{n=1}^{\infty} \frac{x^{2^{n-1}}}{x^{2^n} - 1} = \begin{cases} 1/(x-1), & \text{if } |x| > 1, \\ x/(x-1), & \text{if } |x| < 1. \end{cases}$$

Equation (1) follows at once from (3) in the *special* case when $x = 2$.

Remark: The general summation formulas (2) and (3) are attributed to De Morgan (1806-1871) and Tannery (1848-1910), respectively, by Bromwich (see [1], Example 15, p. 24; Example 24, p. 273). In fact, (3) has appeared in *numerous* books and tables.

1. T. J. I'A. Bromwich. *An Introduction to the Theory of Infinite Series.* 2nd ed. London: Macmillan, 1926.

Also solved by P. Bruckman, D. Carothers, C. Georghiou, W. Janous, R. Andre-Jaennin, C. Long, H.-J. Seiffert, P. Tzermias, and the proposer.

Editorial Note: The editor wishes to apologize to Paul Bruckman for the omission of his name in the solution of H-409. The editor would like anyone with identities relating to H-409 to submit them to John Turner, University of Waikato, New Zealand, for his judgment as to the awarding of the $25 prize.

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Announcement

FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 30-August 3, 1990

Department of Mathematics and Computer Science

Wake Forest University

Winston-Salem, North Carolina 27109

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CALL FOR PAPERS

The FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Wake Forest University, Winston-Salem, N.C., from July 30 to August 3, 1990. This Conference is sponsored jointly by the Fibonacci Association and Wake Forest University.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts were to be submitted by March 15, 1990. However, there is still some room on the schedule for speakers. Submit abstracts as soon as possible. Manuscripts are due by May 30, 1990. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

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