# PARTIAL ORDERS AND THE FIBONACCI NUMBERS

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## Introduction

We present an approach to the Fibonacci numbers by considering finite partially ordered sets (posets). The $n^{\text{th}}$ Fibonacci number, $F_n$ can be interpreted as the number of ideals in a very simple poset, usually called a *fence*.

The purpose of this note is *not* to prove new theorems about the sequence $\{F_n\}$. However, we wish to demonstrate that the approach has several advantages. By attaching to each Fibonacci number a geometrical object, the number gets an additional dimension, that might be of value in proving identities for the Fibonacci numbers.

While, in general, it may be difficult to count the number of ideals in a poset, the simple structure of a fence enables one to calculate the number of ideals in several different ways.

Even the simple partition of the ideals in a fence into two classes, those that contain a given element $x$, and those that do not contain $x$, can be used to show properties of the Fibonacci numbers that usually are verified by an inductive proof. This may, in some cases, add to our understanding of "why" the proof is valid.

Another advantage is that, after having established that $F_n$ is the number of ideals in a fence with $n$ elements, we have at our disposal theorems from the general theory of posets, see for instance [2].

## Preliminaries

Our terminology on posets is, with a few exceptions, standard, and we refer to for instance Birkhoff [1], but for the convenience of the reader, we define briefly the basic concepts.

We let $[n]$ denote the set $\{1, \ldots, n\}$.

In this paper a partially ordered set (*poset*) is a *finite* set equipped with a relation $\geq$ that is reflexive, antisymmetric, and transitive.

An *ideal* in a poset $P$ is a subset $I$ of $P$ such that, for any $x \in P$ and any $y \in I$, if $x \geq y$ then $x \in I$. Both $\emptyset$ and $P$ are ideals in $P$. Actually, an ideal in the present paper is usually called an *upper* ideal, *dual* ideal, or *filter*.

For any poset $P$, $Id(P)$ denotes the number of ideals in $P$. Moreover, $Id(x)$, $Id(x \;\&\; y)$, and $Id(x \;\&\; \neg y)$ denote the number of ideals (in $P$) that contain $x$, contain $x$ and $y$, contain $x$ but not $y$, respectively.

Given a subset $A$ of a poset $P$, let $A^*$ denote the set of elements $x \in P$ such that $x \geq a$ for some $a \in A$, and $A_*$ denotes the elements $x \in P$ such that $a \geq x$ for some $a \in A$.

Any subset $A$ of a poset $P$, may be considered as a poset in itself with the inherited relations from the set $P$. Hence, $Id(A)$ denotes the number of ideals in the *poset* $A$. This should not be confused with the earlier definitions of $Id(x)$, $Id(x \;\&\; y)$, etc.

The elements $x$ and $y$ in a poset $P$ are *path connected* if there exists a sequence of elements $x_1, \ldots, x_n$ in $P$ such that $x_1 = x$, $x_n = y$, and $x_i$ and $x_{i+1}$ are comparable for each $1 \leq i \leq n - 1$. Two subsets $A$ and $B$ of a poset are *separated* if $x$ and $y$ are *not* path connected for any $x \in A$ and $y \in B$.

The *fence* $\Gamma_n$ with $n$ elements is the poset

$$\Gamma_n = \{x_1 \geq x_2 \leq x_3 \geq \cdots \leq (\text{or} \geq)\ x_n\}.$$

Let $\Gamma_0$ refer to the empty fence, with one ideal only.

A fence can be pictured as a lattice path; we show $\Gamma_5$ in Figure 1.

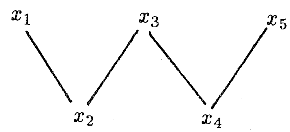


FIGURE 1: $\Gamma_5$

The following observation, whose simple proof is omitted, will be found to be very useful.

*Lemma 1:* Let $A$ be a subset of the poset $P$. Then:

1. The number of ideals in $P$ that contain $A$ equals $Id(P - A^*)$.
2. The number of ideals in $P$ that are disjoint with $A$ equals $Id(P - A_*)$.
3. If $P = A \cup B$, where $A$ and $B$ are separated subsets of $P$, then $Id(P) = Id(A)Id(B)$.

As an illustration of Lemma 1, we shall find $Id(x_3)$ and $Id(\neg x_3)$ for the fence $\Gamma_5$. In order to find $Id(x_3)$, Lemma 1.1 says that one shall erase all $y$ such that $y \geq x_3$, and find the number of ideals in the remaining poset. In this case, we only erase $x_3$ itself, and are left with a poset consisting of two separated parts, each being isomorphic to $\Gamma_2$. Hence, by Lemma 1.3 it follows that $Id(x_3) = Id^2(\Gamma_2)$.

In order to find $Id(\neg x_3)$, one must erase $\{x_3\}_* = \{x_2, x_3, x_4\}$. One is left with two separated copies of $\Gamma_1$; thus, $Id(\neg x_3) = Id^2(\Gamma_1)$. Hence,

$$Id(\Gamma_5) = Id^2(\Gamma_2) + Id^2(\Gamma_1).$$

## Ideals in a Fence

Let $F_0 = 1$, $F_1 = 2$, $F_2 = 3$, etc., refer to the Fibonacci numbers, and $\Gamma_n$ to the fence of cardinality $n$.

*Theorem 1:* $Id(\Gamma_n) = F_n$ for $n = 0, 1, 2, \ldots$ .

*Proof:* By definition $Id(\Gamma_0) = 1$, and trivially $Id(\Gamma_1) = 2$. We shall show that

$$Id(\Gamma_n) = Id(\Gamma_{n-1}) + Id(\Gamma_{n-2}) \text{ for } n \geq 2.$$

In general,

$$Id(\Gamma_n) = Id(x_n) + Id(\neg x_n).$$

If $n$ is even, it follows from Lemma 1 that

$$Id(x_n) = Id(\Gamma_{n-2}) \quad \text{and} \quad Id(\neg x_n) = Id(\Gamma_{n-1}),$$

and if $n$ is odd, Lemma 1 yields that

$$Id(x_n) = Id(\Gamma_{n-1}) \quad \text{and} \quad Id(\neg x_n) = Id(\Gamma_{n-2}).$$

This proves Theorem 1.

We shall consider a few simple applications of Theorem 1.

*Corollary 1:* $F_n = F_{i-1}F_{n-i} + F_{i-2}F_{n-i-1}$ for $2 \leq i < n$.

*Proof:* Follows from Theorem 1, Lemma 1, and the identity

$$Id(\Gamma_n) = Id(x_i) + Id(\neg x_i).$$

In the remainder of this note we simplify our notation by letting the nodes of $\Gamma_n$ be denoted by 1, ..., $n$ instead of $x_1, \ldots, x_n$.

*Corollary 2:*

$$F_{2n-1} = \#\{(a_1, \ldots, a_k) \mid a_i \text{ is odd and } a_i \geq 1 \text{ and } a_1 + \cdots + a_k = 2n + 1\}.$$

*Proof:* A subset $X$ of $[n]$ can uniquely be given by an odd (i.e., $k$ = odd) tuple $(a_1, \ldots, a_k)$ of positive integers whose sum equals $n + 2$. To such a tuple we assign the set $X$ defined by: $a_1$ is the smallest number belonging to $X$, $a_1 + a_2$ is the smallest number greater than $a_1$ that does not belong to $X$, $a_1 + a_2 + a_3$ is the smallest number after $a_1 + a_2$ that belongs to $X$, etc.

The following example illustrates the correspondence. Let $n = 11$ and let $(a_1, \ldots, a_5) = (2, 3, 2, 2, 4)$. This vector corresponds with the set {2, 3, 4, 7, 8}.

It is easily seen that by this correspondence, the set corresponding to a vector $(a_1, \ldots, a_k)$ is an ideal in $\Gamma_{2n-1}$ iff each $a_i$ is an odd integer.

This proves Corollary 2.

*Corollary 3:* $F_{2n-1} = \sum_{i=0}^{n} \binom{n+i}{2i}$

*Proof:* By Corollary 2, $F_{2n-1}$ equals the number of tuples $(a_1, \ldots, a_k)$ of odd positive integers whose sum is $2n + 1$. Put $a_j = 2b_j - 1$, and since $k$ is odd, there exists an integer $i$ such that $k = 2i + 1$. One derives the condition

$$b_1 + \cdots + b_{2i+1} = n + i + 1$$

and since

$$\#\{(c_1, \ldots, c_i) \mid c_i \geq 1 \text{ and } c_1 + \cdots + c_i = m\} = \binom{m-1}{i-1}.$$

Corollary 3 follows.

Finally, let us add that many more identities can be shown in this simple manner.

A slightly more complicated application is achieved by defining an equivalence relation on $\Gamma_{2n-1}$ by declaring two ideals to be equivalent if they contain the same *odd* numbers in $[2n - 1]$. Counting the number of ideals in each equivalence class leads to the following identity, whose proof is left to the reader.

$$F_{2n-1} = 1 + \sum \binom{s-1}{k-1}\binom{n+1-s}{k} 2^{s-k},$$

where the sum is over all $(s, k)$ such that $s \geq k \geq 1$ and $s + k \leq n + 1$.

## References

1. G. Birkhoff. *Lattice Theory*. AMS, Coll. Publ. Vol. XXV, 1973.
2. I. Rival, ed. *Ordered Sets*. Dordrecht: D. Reidel, 1982.

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