GENERALIZED FIBONACCI POLYNOMIALS AND THE FUNCTIONAL ITERATION OF RATIONAL FUNCTIONS OF DEGREE ONE

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1. Introduction

One of the great advances in mathematics recently has been in the analysis of nonlinear dynamical systems. In this paper we will study the properties of a set of polynomials in two variables using techinques from nonlinear dynamic theory. These polynomials are variants of the class of generalized Fibonacci polynomials (see, for example, [7]) defined by

$$P_0(z_1, z_2) = 0, \quad P_1(z_1, z_2) = 1,$$
$$P_{n+1}(z_1, z_2) = (1 - z_1)P_n(z_1, z_2) - (z_2 - z_1)P_{n-1}(z_1, z_2), \; n \geq 1.$$

The results derived here are not new in the sense that they can be proven from existing work on generalized Fibonacci polynomials but the approach is entirely novel in that it provides a link between the analysis of generalized Fibonacci numbers and the theory of dynamical systems via the iteration of rational functions of degree one.

Fundamental to the concept of the analysis of nonlinear dynamical systems is the functional iteration of the form

$$x_{n+1} = f(\lambda, x_n), \tag{1}$$

where λ is a parameter that can be varied. In this paper we will consider the iterative behavior of the general rational function of degree one given by

$$f(k, \lambda_1, \lambda_2, x) = k\,\frac{1 - \lambda_1 x}{1 - \lambda_2 x}, \tag{2}$$

where k, λ_1, and λ_2 can be complex, and relate these iterations to a family of polynomials, defined in two variables by

$$P_0(z_1, z_2) = 0, \quad P_1(z_1, z_2) = 1, \tag{3}$$
$$P_{n+1}(z_1, z_2) = (1 - z_1)P_n(z_1, z_2) - (z_2 - z_1)P_{n-1}(z_1, z_2), \; n \geq 1.$$

We will also consider as a special example the case when $k = 1$ and $\lambda_1 = 0$, so that

$$f(\lambda, x) = \frac{1}{1 - \lambda x}, \tag{4}$$

and relate the iterations of this class of functions to a family of polynomials defined by

$$P_0(z) = 0,\; P_1(z) = 1,\; P_{n+1}(z) = P_n(z) - zP_{n-1}(z),\; n \geq 1. \tag{5}$$

We note that in our terminology $P_n(0, z) = P_n(z)$. The polynomials presented in (3) and (5) are in fact variants of two well-known classes of polynomials known as generalized Fibonacci polynomials and Fibonacci polynomials, respectively.

In Section 2 we will present a review of some of the known results concerning generalized Fibonacci polynomials and show that they can be generalized to the polynomials defined in (3) and (5). The analysis in Section 3 will prove some of these results anew but using a completely different approach. This approach is based on the concept of topological conjugacy. Two maps $f: A \to A$

and $g: B \to B$ are said to be topologically conjugate if there exists a homeomorphism $h: A \to B$ such that

$$h \circ f = g \circ h. \tag{6}$$

Topologically conjugate maps are equivalent in terms of their dynamics (see, for example [4]). Now, if g is the function μz, then (6) is called the Schröder Functional Equation (SFE). It is well known (see, for example, [1]) that, if f is a rational function of degree two or more, then the SFE does not have a solution if μ is a root of unity. On the other hand, Siegel [11] has shown that, if $\mu = e^{2\pi i \alpha}$, where α is irrational, then the SFE has a solution if there exist a, $b > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{a}{q^b}, \quad \forall p,\ q \in \mathbf{Z}.$$

This condition is satisfied for a set of μ of full measure on the unit circle. In this paper we will make use of the well-known fact that f, given by (2), is topologically conjugate to μx. Hence, the dynamics of f and μx are equivalent and the zeros of the generalized Fibonacci polynomials can be related to the roots of unity.

2. Generalized Fibonacci Polynomials

Although Fibonacci polynomials have been studied for well over a century, there was initially no common agreement on how to define this class of polynomials. For example, Catalan [3] defined them by

$$F_0(z) = 0,\ F_1(z) = 1,\ F_{n+1}(z) = zF_n(z) + F_{n-1}(z),\ n \ge 1,$$

while Jacobsthal [9] defined them by

$$F_0(z) = 0,\ F_1(z) = 1,\ F_{n+1}(z) = F_n(z) + zF_{n-1}(z),\ n \ge 1,$$

and Byrd [2] by

$$F_0(z) = 0,\ F_1(z) = 1,\ F_{n+1}(z) = 2zF_n(z) + F_{n-1}(z),\ n \ge 1.$$

However, the general consensus (see [6], for example) is that the class of Fibonacci polynomials is defined by

$$F_0(z) = 0,\ F_1(z) = 1,\ F_{n+1}(z) = zF_n(z) + F_{n-1}(z),\ n \ge 1. \tag{7}$$

It is easy to obtain a simple closed expression for these polynomials in terms of trigonometric functions (see [6], for example) and hence show that the zeros of F_n are given by

$$2i \cos\frac{k\pi}{n},\ k = 1,\ \ldots,\ n - 1.$$

In addition, it is easy to show

$$F_n(z) = \sum_{j=0}^{p} \binom{n-1-j}{j} z^{n-2j-1},\ p = \left[\frac{n-1}{2}\right]. \tag{8}$$

Horadam [8] has considered generalized sequences of Fibonacci numbers given by

$$w_0 = a,\ w_1 = b,\ w_{n+1} = pw_n - qw_{n-1},\ n \ge 1,$$

where w_n is a function of a, b, p, and q, and obtained closed expressions for many special classes of w_n. The case in which $a = 0$, $b = 1$ so that

$$\begin{aligned} &F_0(z_1, z_2) = 0,\quad F_1(z_1, z_2) = 1, \\ &F_{n+1}(z_1, z_2) = z_1F_n(z_1, z_2) + z_2F_{n-1}(z_1, z_2),\ n \ge 1 \end{aligned} \tag{9}$$

is now known as the family of generalized Fibonacci polynomials. The properties of these polynomials have been studied extensively by Hoggatt & Long [7], which builds on the earlier work of Webb & Parberry [12] who consider the divisibility properties of Fibonacci polynomials.

In particular, Hoggatt & Long [7] show that

$$(10)\qquad F_n(z_1, z_2) = \sum_{j=0}^{p}\binom{n-1-j}{j} z_1^{n-2j-1} z_2^j, \quad p = \left[\frac{n-1}{2}\right],$$

and that $F_n(z_1, z_2) = 0$ iff

$$z_1 = 2i\sqrt{z_2}\cos\frac{k\pi}{n}, \quad k = 1, \ldots, n-1.$$

Furthermore, they show that, for $m \geq 2$, $F_m | F_n$ iff $m|n$ and that F_n is irreducible over the rationals iff A is prime. A consequence of this is, if $n_1, \ldots, n_\ell$ are the factors of n, then all the zeros of $F_{n_1}, \ldots, F_{n_\ell}$ are zeros of F_n.

This work has been generalized by Kimberling [10] who shows that each generalized Fibonacci polynomial F_n has one and only one irreducible factor that is not a factor of F_k for any $k < n$, which is called the n^{th} Fibonacci cyclotomic polynomial $G_n(z_1, z_2)$. Kimberling shows

$$F_n(z_1, z_2) = \prod_{d|n} G_n(z_1, z_2).$$

The polynomials defined in (3) and (5), which will prove significant when analyzing the behavior of the iteration of rational functions of degree one, can easily be related to generalized Fibonacci polynomials and Fibonacci polynomials. In fact, comparing (3) and (9), we see

$$(11)\qquad P_n(z_1, z_2) = F_n(1 - z_1, z_1 - z_2),$$

while

$$(12)\qquad P_n\left(\frac{-1}{x^2}\right) = \frac{F_n(x)}{x^{n-1}}$$

or

$$(13)\qquad P_n(z) = \frac{F_n\left(\frac{i}{\sqrt{z}}\right)}{\left(\frac{i}{\sqrt{z}}\right)^{n-1}}.$$

This can be seen by substituting (12) into (5) and noting that (7) results. Consequently, it is trivial to show

$$P_n(z) = \sum_{j=0}^{p}(-1)^j\binom{n-1-j}{j} z^j, \quad p = \left[\frac{n-1}{2}\right],$$

while

$$(14)\qquad P_n(z_1, z_2) = \sum_{j=0}^{p}\binom{n-1-j}{j}(1 - z_1)^{n-2j-1}(z_1 - z_2)^j, \quad p = \left[\frac{n-1}{2}\right].$$

In addition, the zeros of $P_n(z_1, z_2)$ and P_n can be found from (11) and (13). Thus, the zeros of P_n are simple and given by

$$\frac{1}{4}\sec^2\frac{k\pi}{n}, \quad k = 1, \ldots, n-1,$$

so that all zeros are real distinct and lie in the interval $(1/4, \infty)$. Similarly, if $z_1 \neq 1$, then $P\ (z_1, z_2) = 0$ iff

$$(15)\qquad z_2 = z_1 + (1 - z_1)^2\frac{1}{4}\sec^2\frac{k\pi}{n}, \quad k = 1, \ldots, n-1,$$

where all the roots in this set are simple, so that if $n = 2p + 1$ there are p distinct zeros while if $n = 2p$ there are $p - 1$. On the other hand if $z_1 = 1$, then (14) implies

(16) $\quad P_{2n}(1, z_2) = 0, \; n = 1, 2, \ldots,$

while it can easily be seen from (14) that

(17) $$P_n\left(z_1, \frac{(1 + z_1)^2}{8}\right) = n\left(\frac{1 - z_1}{2}\right)^{n-1}.$$

We also note that a formula for P_n can be derived by considering the characteristic polynomial associated with (3) given by

$$x^2 - (1 - z_1)x + z_2 - z_1 = 0.$$

The roots of this equation are

$$\theta_{\pm} = \frac{1 - z_1 \pm \sqrt{(1 + z_1)^2 - 4z_2}}{2}$$

and so it is easily seen that

(18) $$P_n(z_1, z_2) = \left(\frac{\theta_+^n - \theta_-^n}{\theta_+ - \theta_-}\right).$$

In the next section we will show that some of the above results can be proved by noting the behavior of the iterations of rational functions of degree one. For ease of notation we will henceforth refer to the polynomials $P_n(z_1, z_2)$ as the Shifted Generalized Fibonacci Polynomials (SGFP).

3. Functional Iteration

Consider the iteration scheme given by (1) where f is as in (2). We will denote the iterations of $\{x, x_1, x_2, \ldots, x_n, \ldots\}$ by

$$\{f^{(k)}(x);\; k = 0, 1, \ldots\}.$$

The following result gives the value of x_n after n iterations.

Lemma 1: Let $z_1 = k\lambda_1$, $z_2 = k\lambda_2$, and $P_n(z_1, z_2)$ represent the n^{th} shifted generalized Fibonacci polynomial then

$$f^{(n)}(x) = \frac{kP_n(z_1, z_2) - x(P_n(z_1, z_2) - P_{n+1}(z_1, z_2))}{P_{n+1}(z_1, z_2) + z_1P_n(z_1, z_2) - x\lambda_2P_n(z_1, z_2)}.$$

Proof: The proof is by induction. From (2),

$$f^{(2)}(x) = k\,\frac{1 - \lambda_1 k\dfrac{1 - \lambda_1 x}{1 - \lambda_2 x}}{1 - \lambda_2 k\dfrac{1 - \lambda_1 x}{1 - \lambda_2 x}} = \frac{k(1 - z_1) - x(z_2 - z_1^2)}{1 - z_2 - \lambda_2 x(1 - z_1)}$$

$$= \frac{kP_2(z_1, z_2) - x(P_2(z_1, z_2) - P_3(z_1, z_2))}{P_3(z_1, z_2) + z_1P_2(z_1, z_2) - \lambda_2 xP_2(z_1, z_2)},$$

where $z_1 = k\lambda_1$, $z_2 = k\lambda_2$. Now,

$$f^{(n+1)}(x) = f^{(n)}(f(x)) = \frac{kP_n - k\dfrac{1 - \lambda_1 x}{1 - \lambda_2 x}(P_n - P_{n+1})}{P_{n+1} + z_1P_n - z_2\dfrac{1 - \lambda_1 x}{1 - \lambda_2 x}P_n}$$

$$= \frac{kP_{n+1} - x(z_2P_n - z_1(P_n - P_{n+1}))}{P_{n+1} + z_1P_n - z_2P_n - \lambda_2 xP_{n+1}} = \frac{kP_{n+1} - x(P_{n+1} - P_{n+2})}{P_{n+2} + z_1P_{n+1} - \lambda_2 xP_{n+1}},$$

by (3), and the lemma is proved.

From Lemma 1, it can be seen that

$$f^{(n)}(x) = x + P_n(z_1, z_2)\frac{\lambda_2 x^2 - (1 + z_1)x + k}{P_{n+1}(z_1, z_2) + z_1 P_n(z_1, z_2) - \lambda_2 x P_n(z_1, z_2)}, \tag{19}$$

so that x is a fixed point of $f^{(n)}$ iff

$$P_n(z_1, z_2) = 0, \quad \text{or} \quad \lambda_2 x^2 - (1 + z_1)x + k = 0. \tag{20}$$

Thus, it can be seen that, if

$$P_n(z_1, z_2) = 0,$$

then f is periodic of order n no matter what the starting value [or, equivalently, $f^{(n)}(x)$ is the identity function]. From this, we deduce that the result in [7] about the common zeros of generalized Fibonacci polynomials is a direct consequence of (20). For, if N is a multiple of n, and z_1 and z_2 are chosen so that $P_n(z_1, z_2) = 0$, then f will be periodic of order n for any starting value. But f will also be periodic of order N, and so from (19), $P_N(z_1, z_2) = 0$. Thus, $P_n | P_N$ iff $n | N$.

The above property is due to the well-known fact that the map given by (2) is topologically conjugate to the map μz by a Möbius transformation (see, for example, [4]). Consequently, if the function $g(z) = \mu z$ is iterated, then g will be periodic of order n for any initial guess if $\mu^n - 1 = 0$; hence, the zeros of the shifted generalized Fibonacci polynomials are related to the n^{th} roots of unity.

Some simple analysis gives the relationship between μ and (2) as

$$\mu = \frac{1 - 2z_2 + z_1^2 \pm (1 - z_1)\sqrt{(1 + z_1)^2 - 4z_2}}{2(z_2 - z_1)} = \frac{\theta_\pm}{\theta_\mp}, \tag{21}$$

where $z_1 = k\lambda_1$, $z_2 = k\lambda_2$. This can also be written as

$$\mu^2 - \mu\left(\frac{1 - 2z_2 + z_1^2}{z_2 - z_1}\right) + 1 = 0. \tag{22}$$

Hence, from (18) and (21), we have

$$\mu^n - 1 = \frac{\theta_\pm^n - \theta_\mp^n}{\theta_\mp^n} = \frac{\theta_\pm - \theta_\mp}{\theta_\mp^n} P_n(z_1, z_2) = \frac{\sqrt{(1 + z_1)^2 - 4z_2}}{\theta_\mp^n} P_n(z_1, z_2). \tag{23}$$

Now the dynamics of g and f are equivalent (see, for example, [4]). If $|\mu| < 1$, then the iterations of g converge to 0 for any starting value while, if $|\mu| > 1$, the iterations converge to infinity for any starting value apart from 0. On the other hand, if $|\mu| = 1$, there are two possibilities: if μ is an n^{th} root of unity, the iterations of g are periodic of order n for any starting value, so that $g^{(n)}$ is the identity function while, if $\mu^n \neq 1$, then the iterations of $g(x)$ wander chaotically on the unit disk of radius x taking on all possible values. Thus, the relationship between the zeros of unity and the zeros of P_n are obtained from (22) and (23) by noting the following:

(i) $\mu = 1$ corresponds to $(1 + z_1)^2 = 4z_2$, so that from (17) and Lemma 1,

$$f^{(n)}(x) = \frac{k - x\left(1 - \frac{1}{2}\left(1 + \frac{1}{n}\right)(1 - z_1)\right)}{\frac{1}{2}\left(1 + \frac{1}{n}\right)(1 - z_1) + z_1 - \frac{x}{4k}(1 + z_1)^2} \to \frac{2k}{1 + z_1} \text{ as } n \to \infty.$$

(ii) $\mu = -1$, which is equivalent to $\mu^n = 1$ for n even, corresponds [by (16) and (22)] to $z_1 = 1$. In this case f is periodic of order 2 for any starting value.

(iii) $\mu^n = 1$, with $\mu \notin \{1, -1\}$, implies [from (22) and (23)] that the zeros of P_n are

$$z_2 = \frac{\mu}{(\mu + 1)^2}(1 - z_1)^2 + z_1. \tag{24}$$

For these values of z_1, $f^{(n)}$ is the identity function.

Thus, in conclusion, we have seen that by iterating the general rational function of degree one and noting that the dynamics of this function are the same as that of the function μz, we have obtained relationships between the zeros of generalized Fibonacci polynomials and the n^{th} roots of unity. These results are not new but the proofs are and they rely upon obtaining a general formula for the n^{th} iteration of a rational function of degree one in terms of a set of polynomials called Shifted Generalized Fibonacci Polynomials. Thus, we have related the study of Fibonacci theory to the iteration of the general rational function of degree one.

With respect to the mathematics of the iteration of nonlinear functions, since it is known that the Schröder Functional Equation has no solution for rational functions of degree 2 or more when μ is an n^{th} root of unity, we have, in this paper, essentially characterized the dynamics of all rational functions that satisfy the SFE when μ is a root of unity. Finally, in this paper we have obtained results about the nature of the zeros of a new class of polynomials by iterating an appropriate class of functions and this technique may well be generalizable.

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