PALINDROMIC DIFFERENCES

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Let x denote a positive integer, written in the ordinary denary form, and define its palindromic inverse x' to be the integer obtained from x by writing its digits in reverse order. We ignore leading zeros so that both 1234 and 12340 have palindromic inverse 4321. A number is called a palindrome if $x = x'$. Similar definitions apply to bases other than 10.

A notorious problem concerns palindromic sums [3]. From any starting point x_1 we form a sequence inductively by $x_{k+1} = x_k + x_k'$, and the question is whether one always arrives at a palindrome. A negative answer is conjectured, and specifically that for $x_1 = 196$ a palindrome is never reached. Although this problem is unsolved, the conjecture is known to be correct for base 2 [2]. The problem, however, is somewhat artificial since the property of being a palindrome will not persist throughout the iteration even if ever attained. We consider here the problem of taking palindromic differences; starting with x_1, define

$$x_{k+1} = |x_k - x_k'|$$

inductively. In this case, if x_k were a palindrome, all its successors would vanish, and the first question that arises is whether this always occurs. This problem has been considered previously (see [1], [4], [5]).

Clearly, if x_1 has only one digit, then $x_2 = 0$, and if x_1 has two digits, then x_2 will have at most two digits and be divisible by 9. If $x_2 = 9$ or 99, then $x_3 = 0$, whereas all other cases do eventually reach zero, as the sequence 90, 81, 63, 27, 45, 9, 0 shows, for this sequence together with all palindromic inverses contains all integers of no more than two digits divisible by 9. The same reasoning applies to three-digit numbers, for then x_2 will be divisible by 99, and the sequence 990, 891, 693, 297, 495, 99, 0 shows just as before that, for any x_1 under 1000, the process leads to zero in the end. As we shall see presently, the close connection between the behavior for two- and three-digit numbers is not mere coincidence.

Given an x_1 having n digits, it is not necessarily true that $x_2 < x_1$, but certainly x_2 has n or fewer digits. Accordingly, from any starting point x_1 of digit length n one of two things must happen; either in the sequence of iterates we find one with fewer than n digits, which property will then persist, or else the sequence becomes periodic eventually with all the numbers in the period having n digits. Within a period, the period-length p, is the number of iterations required to return to the starting point. We have already seen that there are no periods with $0 < n < 4$. However, there is a period with $n = 4$, $p = 2$, with $x_1 = 2178$, $x_2 = 6534$. So there are nontrivial periods. We seek to determine for each n, all possible periods; alternately, we might desire to find all possible p.

It is easily seen that $p = 1$ cannot occur except for $x_1 = 0$, for it would require $x_2 = x_1$ and so $x_1' = 2x_1$. Suppose then that the first and last digits of x_1 were a and b, respectively. Then we should find that $b = 2a$ or $2a + 1$ and also that $a \equiv 2b \pmod{10}$, which cannot hold simultaneously. [Incidentally, it can be shown that if instead of base 10 we consider base β the same result holds if $\beta = 2$ or if $\beta \equiv 1 \pmod 3$. However, in other cases, there are nontrivial periods with $p = 1$, e.g., $x = ab$ with

$$a = (\beta - 2)/3,\ b = (2\beta - 1)/3 \quad \text{if } \beta \equiv 2 \pmod 3,$$

and $x = abcd$ with

$$a = \beta/3,\ b = (\beta - 3)/3,\ c = (2\beta - 3)/3,\ d = 2\beta/3 \quad \text{if } \beta \equiv 0 \pmod 3.$$

We shall, however, concentrate on the denary case in the sequel.

We observed before a connection between the behavior of three-digit numbers and that of two-digit numbers, and we now use this to dispose of the case in which n is odd. Suppose that we have a period in which $n = 2m + 1$ is odd, and let $x_1 = a_0 a_1 \ldots a_{2m-1} a_{2m}$ be any number in any period with digit length n. Then x_2 is the modulus of the difference

$$\begin{array}{l} a_0\ a_1 \quad \ldots\ a_m\ \ldots\ a_{2m-1}a_{2m}\ - \\ a_{2m}a_{2m-1}\ \ldots\ a_m\ \ldots\ a_1 \quad a_0 \end{array}$$

and since the two middle digits coincide, the middle digit of the difference will be 9 or 0 accordingly as there is or there is not a carry in the middle of the subtraction. Hence, for every number in such a period the middle digit will be 0 or 9, and moreover, were this digit to be removed in all cases, we should obtain a period with the same p but with n reduced by 1. Conversely, all periods with n odd can be obtained from exactly similar ones with n one less by the insertion of a suitable middle digit 0 or 9; thus, the period 2178, 6534, 2178 leads to 21978, 65934, 21978. In fact, we can produce a period with n one larger still by doubling this middle digit and, of course, the process can be carried on indefinitely. We call a period *old* if it is derived in this way from one with smaller n, and we shall from now onward concentrate on finding the new periods; since all new periods have n even, we shall write $n = 2m$.

Much of what follows was obtained by computation, and economy soon becomes a major consideration. At first sight, it might appear that to find all periods of digit length $2m$ it might be necessary to consider all $9 \cdot 10^{2m-1}$ possible n-digit numbers and their iterates to find all possible periods. Such a procedure would be extremely wasteful, for all the integers in a period are themselves iterates, and there are far fewer of these. For suppose that $x_1 = a_0 a_1 \ldots a_{n-1}$ and without loss of generality that $x_1 < x_1'$. Then

$$x_2 = \sum_{r=0}^{m-1} A_r (10^{n-r-1} - 10^r),$$

where $A_r = a_{n-r-1} - a_r$. Since x_2 has n digits (and not less), it is easily seen that this requires

$$1 \le A_0 \le 9 \quad \text{and} \quad -9 \le A_r \le 9, \quad r = 1, 2, \ldots, m - 1.$$

Secondly, the observation that second iterates cannot have $A_0 = 9$ reduces the number of cases to be considered to $8 \cdot 19^{m-1}$. Despite this reduction and some other refinements, the number of cases still grows exponentially with n, which soon makes complete computation impossible.

We shall represent the iterate x, a number of $2m$ digits, by the corresponding A's in the canonical form $\{A_0, A_1, \ldots, A_{m-1}\}$ where it is to be understood that A_0 lies between 1 and 8 and the others between -9 and 9. From this, the denary form for x is found by writing

$$A_0 A_1 \ \ldots\ A_{m-1}\ -A_{m-1}\ \ldots\ -A_1\ -A_0$$

where, of course, some of the numbers will be negative. To deal with this, we start at the right, and whenever we encounter a negative number add 10 to it and subtract 1 from its predecessor in the usual "borrow and carry" fashion, familiar from elementary arithmetic. The successor is then easily calculated in the same canonical form and the process repeated, in a manner eminently suitable for computation.

It will be clear that if $A_{m-1} = 0$, then in the denary form the number will have its two middle digits both 0 or both 9, and its successor will also have $A_{m-1} = 0$; such a number cannot appear in a new period, and so can be ignored in a search for new periods. At first this appears to produce only a small saving in the computation, a factor of 18/19, but this is not so, for we can ignore any x_1 any of whose *iterates* has $A_{m-1} = 0$, and this observation saves a very large proportion of the time required to compute the periods.

Since we now assume that $A_{m-1} \neq 0$, we can associate with each number x of digit length $2m$ in a new period, the rational number $\mu = \sum A_r \cdot 19^{-r}$ whose denominator is precisely 19^{m-1}, and conversely, each such μ yields a unique x. Within each period we call that x the *first* in the period if the corresponding μ is the least μ of any x in the period. It clearly suffices to find all the first numbers in the periods.

For any r with $0 \le r < m - 1$, we write

$$x_1 = \{A_0, A_1, \ldots, A_r, +\}$$

or

$$x_1 = \{A_0, A_1, \ldots, A_r, -\}$$

according as the first nonvanishing integer in the sequence $A_{r+1}, \ldots, A_{m-1}$ is positive or negative. The utility of this lies in the fact that if

$$x_2 = \{B_0, B_1, \ldots, B_{m-1}\},$$

then $B_0, B_1, \ldots, B_r$ depend only upon $A_0, A_1, \ldots, A_r$ and the value + or - and not on the *actual* values of $A_{r+1}, \ldots, A_{m-1}$. Using this fact, we see that no period contains any element $\{5, +\}$, for the successor would have $B_0 = 0$. Furthermore, no period has $\{4, +\}$ as its first element, for the successor would have $B_0 = 2$, contradicting the assumption that $\{4, +\}$ came first in the period. In this way, we can write a program to determine whether any period could start with $(A_0, A_1, \ldots, A_r, \varepsilon)$, where $\varepsilon = +$ or $-$, for we can calculate the first $r +$ 1 digits in the canonical form of its successor, then there would be two possible second successors, four possible third successors, and so on. At each stage, we can delete any suggested successor which comes before x_1 and so determine whether we could eventually return to x_1, and if so what is the minimum possible period. For $r = 0$, it is possible to show on the back of an envelope that, for the first element of any period $A_0 = 1$ or 2. For $r = 2$, about 3 minutes on a simple home computer suffice to prove

Result 1: The only period with $m = 2$ starts at $\{2, 2\}$ corresponding to 2178, and for $m > 2$, every new period must start at one of

$$\{1, 0, +\}, \{1, 1, \pm\}, \{1, 2, \pm\}, \{1, 3, \pm\}, \{2, -9, \pm\}, \{2, -8, \pm\},$$
$$\{2, -6, \pm\}, \{2, -5, \pm\}, \{2, -3, -\}, \{2, 0, -\}, \text{ or } \{2, 2, -\}.$$

The same program showed that the only periods with $p = 2$ are $\{2, 2\}$ and possibly more starting at $\{2, 2, -\}$. Use of this fact allows us to find all periods with $p = 2$. Let $\sigma(m)$ denote the number of periods both old and new with $p = 2$ and $n = 2m$. One such is, of course, $\{2, 2, 0, 0, \ldots, 0\}$, but this apart, we must have $x_1 = \{2, 2, -\}$ and so $x_2 = \{6, 6, \ldots\}$. If

$$x_2 = \{6, 6, +\} \quad \text{or} \quad \{6, 6, 0, 0, \ldots, 0\},$$

then

$$x_3 = \{2, 3, \pm\} \quad \text{or} \quad \{2, 2, 0, 0, \ldots, 0\},$$

respectively, and in either case $x_3 \neq x_1$. Thus,

$$x_2 = \{6, 6, -\}.$$

Now consider the number 2199 ... 9978 - x_1. It is easily seen that for some $k \geq 2$ this number has its first k digits zero, its last k digits zero, and a number y, which occupies the middle $2m - 2k$ digits; then

$$x_1 = \{2, 2, 0, \ldots, 0, A_k, \ldots, A_{m-1}\}$$

with $A_k < 0$. Then

$$y_1 = \{-A_k, \ \ldots, \ -A_{m-1}\}.$$

Also, $y_1 < y_1'$, otherwise we should not have $x_2 = \{6, 6, -\}$ and, moreover, y_1 must also be periodic with period dividing 2, and hence equal to 2. Therefore, $y_1 = \{2, 2, \ldots\}$, etc. Conversely, given such a y, we can find a corresponding x_1 of digit length $2m$. Hence,

$$\sigma(m) = 1 + \sigma(1) + \cdots + \sigma(m - 2)$$

and so

$$\sigma(m + 1) = \sigma(m) + \sigma(m - 1).$$

Since $\sigma(1) = 0$ and $\sigma(2) = 1$, it follows that $\sigma(m + 1) = F_m$, the m^{th} Fibonacci number. Also, the number of *old* periods with $p = 2$ and of digit length $2m$ equals $\sigma(m - 1)$; hence, for $m \geq 3$, the number of new periods of digit length $2m$ equals $F_{m-1} - F_{m-2} = F_{m-3}$.

We show next that all periods starting at $\{2, 2, -\}$ have $p = 2$. For, let x_1 be the first element in the period; then $x_2 = \{6, 6, \pm\}$. We cannot have $x_2 = \{6, 6, +\}$, otherwise $x_3 = \{2, 3, \pm\}$ or $\{2, 4, \pm\}$, whence $x_4 = \{2, -\}$ —impossible, since x_1 was assumed to be the first in the period. Thus,

$$x_2 = \{6, 6, -\} \quad \text{and} \quad x_3 = \{2, 2, \pm\}.$$

Again the + sign is impossible, since it would be found that x_5 came before x_1. Thus, we find that, for all k,

$$x_{2k+1} = \{2, 2, -\} \quad \text{and} \quad x_{2k} = \{6, 6, -\}$$

and, accordingly, p must be even. If we now subtract x_1 from 2199 ... 9978, we find that after deleting leading and trailing zeros we obtain either zero or else a number y_1 which also forms part of a periodic sequence with the properties that, for each k,

$$y_{2k+1} < y_{2k+1}' \quad \text{and} \quad y_{2k} > y_{2k}'.$$

It is not very difficult to establish that these conditions also require y_1 to start $\{2, 2, \pm\}$; we omit the details. Hence, all periods starting at $\{2, 2, -\}$ are obtained by the construction above; thus, by induction on m, all have period 2. Summing up, we have

Result 2: Every period with $p = 2$ starts with $\{2, 2, \ldots\}$ and conversely. For given digit-length $2m$ where $m \geq 3$, there are precisely F_{m-1} such distinct periods of which precisely F_{m-3} are new periods, F_k denoting the m^{th} Fibonacci number.

For other values of p, there does not seem to be such a neat description. We have carried out a complete search for $m \leq 8$ and obtained the following

Result 3: For $m \le 8$, the only periods are:

m	p	First x_1	Canonical Form
2	2	2178	2, 2
3	2	219978	2, 2, 0
4	2	21999978	2, 2, 0, 0
	2	21782178	2, 2, -2, -2
	14	11436678	2, -8, -6, 4
5	2	2199999978	2, 2, 0, 0, 0
	2	2178002178	2, 2, -2, -2, 0
	2	2197821978	2, 2, 0, -2, -2
	14	1143996678	2, -8, -6, 4, 0
6	2	219999999978	2, 2, 0, 0, 0, 0
	2	217800002178	2, 2, -2, -2, 0, 0
	2	217821782178	2, 2, -2, -2, 2, 2
	2	219780021978	2, 2, 0, -2, -2, 0
	2	219978219978	2, 2, 0, 0, -2, -2
	12	118722683079	1, 2, -1, -3, 2, 3
	14	114399996678	2, -8, -6, -4, 0, 0
	22	125520874479	1, 2, 5, 5, 2, 1
7	2	eight periods	
	12	one old period	
	14	one old period	
	22	one old period	
8	2	thirteen periods	
	12	one old period	
	14	one old period	
	14	1143667811436678	2, -8, -6, 4, -4, 6, 8, -2
	17	1186781188132188	2, -9, 9, -3, -3, 9, -9, 2
	22	one old period	

It will be observed in the above that certain of the canonical forms of new periods read the same left to right as right to left, e.g., {2, 2} and {1, 2, 5, 5, 2, 1} and that others do so with a change of sign, e.g., {2, 2, -2, -2}. Consider any $x = \{A_0, \ldots, A_{m-1}\}$ in which $A_{m-1} \neq 0$ and define the *dual* of x, $z = \{C_0, \ldots, C_{m-1}\}$ where the A's have been written down back to front and the signs changed throughout if $A_{m-1} < 0$; formally

$$C_r = \operatorname{sgn}\{A_{m-1}\} \cdot A_{m-r-1}, \; 0 \le r \le m - 1.$$

Clearly, performing the operation twice will yield x again, justifying the name "dual." There is one difficulty that arises, for if $A_{m-1} = \pm 1$ and A_{m-2} has opposite sign to A_{m-1}, then $z = \{1, -\}$ and on expansion this fails to have $2m$ digits. We shall deal with this as it occurs. The utility of the definition lies in the following

Lemma: The iterate of the dual equals the dual of the iterate.

Proof; There are two cases depending on the sign of A_{m-1}. We give the proof for $A_{m-1} < 0$, the other case being less transparent but essentially similar. If $x = \{A_0, \ldots, A_{m-1}\}$, then $z = \{-A_{m-1}, \ldots, -A_0\}$. Thus, to find the denary representation for x, we have to perform the "borrow and carry" routine on the expression

$$A_0A_1 \ldots A_{m-1}(-A_{m-1}) \ldots (-A_1)(-A_0),$$

whereas for z we must do the same for

$$(-A_{m-1}) \ldots (-A_1)(-A_0)A_0A_1 \ldots A_{m-1}.$$

Now observing that both A_0 and $-A_{m-1}$ are positive, and the fact that the "first half" of the former expression is identical to the "second half" of the latter and vice-versa, it becomes clear that this property remains intact after the borrowing and carrying; recalling how the iterate is formed from the denary form proves the result.

Now consider any new period which guarantees that $A_{m-1} \neq 0$ for every x in the period. At first sight, the lemma would appear to give a new dual period, obtained by taking duals throughout. There are, however, three reasons why this need not be. In the first place, we might have a period in which x_1 is its own dual, and then by the lemma this property would persist throughout the period. Thus, the dual period does indeed exist, but is identical to the given one. This case can be further subdivided into two cases. If x_1 is its own dual, then we have either $A_r = A_{m-r-1}$ for each r, in which case we call x_1 symmetric, or else $A_r = -A_{m-r-1}$ for each r, in which case x_1 is said to be skew-symmetric. It is not difficult to see that the property of being symmetric or skew-symmetric also persists throughout the iterations and so we also call the respective periods symmetric or skew-symmetric. Both types do exist, as we see in Result 3. The symmetric cases are interesting, and can occur not only if m is even but also with m odd. The skew-symmetric cases, however, are all formed from periods with fewer digits in the following manner. Let

$$x_1 = \{A_0, \ldots, A_{m-1}\}$$

be the first member of any period whatsoever. Then we can obtain a skew-symmetric period with the same p starting at

$$y_1 = \{A_0, \ldots, A_{m-1}, 0, \ldots, 0, -A_{m-1}, \ldots, -A_0\}$$

where the number of zeros written in the middle is arbitrary and can be zero; conversely, any skew-symmetric period is of this form. The symmetric case is entirely different, and although $\{1, 2, 5, 5, 2, 1\}$ belongs to a period, neither $\{1, 2, 5\}$ nor $\{1, 2, 5, 0, 5, 2, 1\}$ does.

A second reason why the dual period may not be interesting is that although x_1 may not be self-dual, it may be the dual of one of its iterates. Thus, if

$$x_1 = \{2, -8, -6, 4\}$$

then

$$x_8 = \{4, -6, -8, 2\}.$$

In such cases it is reasonable to call the period self-dual although the elements themselves are not. It is plain that for all self-dual periods p must be even.

There is a third reason why the dual period may not yield anything interesting. It is possible that one x in a period is of the form we mentioned above with $A_{m-1} = \pm 1$ and A_{m-2} of opposite sign to A_{m-1}, in which case the dual "collapses," in not having the requisite number of digits. This does indeed occur; one example, which may well not be simplest, is the one given in Result 4 below for $p = 9$. It has

$$x_3 = \{4, 3, 4, 7, 0, -3, 9, 1, -6, 2, 2, 3, 2, 6, 7, -9, 6, 8, -4, 1, -4, 9, -2, 1\}.$$

There are some divisibility properties of the x which can occur in a period. Naturally, all are multiples of 9, but the observant reader may have noticed that all the x_1 with $m \le 8$, and indeed all those for $n \le 17$, including those with n odd are multiples of 11. If $n = 2m + 1$ is odd, then any iterate is a multiple of 11 since, if $x_1 = a_0a_1 \ldots a_{2m}$, then

$$\pm x_2 = \sum_{r=0}^{2m} (a_r - a_{2m-r})(10^{2m-r} - 10^r) \equiv 0 \pmod{11}.$$

If $n = 2m$ is even and $x = a_0a_1 \ldots a_{2m-1}$, then

$$x_1 + x_1' = \sum_{r=0}^{2m-r} (a_r + a_{2m-1-r})(10^{2m-1-r} + 10^r) \equiv 0 \pmod{11},$$

and so

$$x_2 = |x_1 - x_1'| \equiv \pm 2x_1 \pmod{11}.$$

Hence, x_1 and x_2 are either both divisible by 11 or neither is. Therefore, in any period either all or none of the numbers are multiples of 11. Let us consider how we might hope to discover periods consisting of nonmultiples of 11. In the first place, if $x_1 = \{A_0, \ldots, A_{m-1}\}$, then

$$x_1 = \sum_{r=0}^{m-1} A_r(10^{2m-r-1} - 10^r) \equiv 2\sum_{r=0}^{m-1} (-1)^{r-1}A_r \pmod{11}.$$

Thus, if x_1 is symmetric and m even, then $11|x_1$. Similarly, if x_1 is skew-symmetric and m odd, but this case is not really interesting, because whatever the parity of m, the property of being divisible by 11 or not is inherited from the shorter period from which x_1 can be formed.

We have seen that $x_2 \equiv \pm 2x_1 \pmod{11}$ and so, if x_1 is not divisible by 11, then

$$x_1 = x_{p+1} \equiv \pm 2^p x_1 \pmod{11}$$

which implies that

$$2^p \equiv \pm 1 \pmod{11},$$

i.e., that 5 divides p. It is not too difficult to show that $p = 5$ will not yield such a value, for if $p = 5$ it can be shown that

$$x_1 = x_6 \equiv 2^5 x_1 \equiv -x_1 \pmod{11}.$$

So in the search for possible periods not divisible by 11, it seems natural to look for numbers with period 10, which are not symmetric with m even nor skew-symmetric. In this way we have been able to find such a period, which is the one listed in Result 4 below; it is self-dual.

From the computational point of view, the existence of such numbers is rather a pity, for had we been able to show that all periods were divisible by 11, the necessary computation to exhaust all possibilities for a given n could have been reduced by a factor of 11.

The next question is, determine for which p periods exist. We have seen that there are none with $p = 1$, but some with $p = 2$, 10, 12, 14, 17, and 22. There is in principle no difficulty, given a suggested p, to search for periods in a systematic way. Suppose that we have reason to think that there might be a period starting at $x_1 = \{A_0, \ldots, A_r, \pm\}$ of period-length p. Then, as mentioned above, we can calculate the 2^{p-1} possible p^{th} successors of x_1 and check whether any one can be $\{A_0, \ldots, A_r, \pm\}$. If not, we can discard this starting point; if yes, then we can increase r by one and look at the 19 possible starting points with the first $r + 1$ entries and the sign given, etc., inductively. Although the task sounds quite formidable, it is actually very efficient at least for small p, apparently more so than a complete search for a given m. In this way, we have been able to show

Result 4: For $p \le 14$, there are no periods with p = 1, 3, 6, or 13. For the other ten values of p, one example each is provided by:

p	Canonical form for x_1
2	{2, 2}
4	{2, -3, 0, -9, 5, -9, 0, -3, 2}
5	{1, 0, 5, 9, 1, 3, -4, 6, 6, -4, 3, 1, 9, 5, 0, 1}
7	{2, -6, 2, 8, -9, 1, -7, 5, 4, 3, 5, 3, 4, 5, -7, 1, -9, 8, 2, -6, 2}
8	{2, -3, 0, -9, 5, -9, -2, 0, -5, 0, 4, 1, 8, 2, -2, -1, 7, 1, -4, -6, -7, -3}
9	{2, -8, -8, -4, 0, 3, 5, 2, -1, -3, 2, 2, -8, -4, 6, -1, 6, 0, 3, 7, 3, 0, 3, -3}
10	{1, 0, 6, -7, 0, -7, -8, 6, -6, -8, 1, 1}
11	{2, -3, -4, 5, -7, -3, 5, 5, -6, 5, -1, 3, -5, -5, 3, -1, 5, -6, 5, 5, -3, -7, 5, -4, -3, 2}
12	{1, 2, -1, -3, 2, 3}
14	{2, -8, -6, 4}

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