SPRINGS OF THE HERMITE POLYNOMIALS

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1. Introduction

The Hermite polynomials, Legendre polynomials, Laguerre polynomials, Gegenbauer polynomials, and Jacobi polynomials belong to the system of classical orthogonal polynomials (see, e.g., [4]). For each class of these polynomials, it is well known that the orthogonal property, differential equation (generalized), Rodrigues representation, and three-term recurrence relation are all equivalent (see, e.g., [4]) in the sense that any one of the above four properties implies the other three.

Throughout this paper we concentrate exclusively on the Hermite polynomials $H_n(x)$. There exist in the literature (see, e.g., [1]-[3], [5], [6], [8]) many starting points for developing the properties of the Hermite polynomials: (i) Hermite differential equation (see, e.g., [6]), (ii) Rodrigues' representation [8], (iii) the simple but beautiful relation [9], given in Arfken ([2], Prob. 13.1.5, p. 718),

(1) $$H_n(x) = (2x - D)^n 1,\ D \equiv d/dx,\ n \geq 0,$$

and (iv) the following generating function (see, e.g., [1]-[3], [5])

(2) $$\exp(2tx - t^2) = \sum_{n=0}^{\infty} H_n(x)t^n/n!$$

Many generating functions exist for the Hermite polynomials (see, e.g., [5]). However, throughout this paper by generating function for $H_n(x)$ we only mean the more familiar one defined by (2). Moreover, we follow the convention that $W^0 = I$, the unit operator, for any operator W. The purpose of this paper is to present the following relation

(3) $$H_n(x) = g^{-1}[2x - D + g^{-1}\{Dg\}]^n g,\ D \equiv d/dx,\ n \geq 0,$$

where $g(x)$ is *any* differentiable function not identically zero, as the spring (starting point) for the starting points. We begin with a derivation of (3) and then show that all properties of the Hermite polynomials and many a beautiful relation follow from it.

2. Spring of Springs

Actually, (3) is a combination of the pure recurrence relation (see, e.g., [5])

(4) $$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x),\ n \geq 1,$$

and the differential recurrence relation (see, e.g., [5])

(5) $$DH_n(x) = 2nH_{n-1}(x),\ n \geq 1,$$

and the results (see, e.g., [5])

(6) $$H_0(x) = 1,$$

(7) $$H_1(x) = 2x.$$

The proof is as follows. Using (4) and (5), we have

(8) $\quad H_{m+1}(x) = 2xH_m(x) - 2mH_{m-1}(x) = (2x - D)H_m(x), \; m \geq 1.$

Moreover, in view of (6) and (7), $H_1(x) = (2x - D)H_0(x)$. Thus,

(9) $\quad H_n(x) = (2x - D)H_{n-1}(x), \; n \geq 1.$

If $g(x)$ is any differentiable function not identically zero, then

(10) $$gH_n(x) = g(2x - D)H_{n-1}(x) = [2x - D + g^{-1}\{Dg\}]\{gH_{n-1}(x)\}, \; n \geq 1.$$

Iteration of (10) yields

(11) $\quad gH_n(x) = [2x - D + g^{-1}\{Dg\}]^n g, \; n \geq 1,$

since $H_0(x) = 1$. However, (11) is also true for $n = 0$. Relation (3) now follows immediately.

The interesting point about (3) is that one need not specify what $g(x)$ is. *Any* differentiable function not identically zero will suffice. Thus, for example, when $g = 1$, we obtain the beautiful relation given in Arfken ([2], Prob. 13.1.5, p. 718):

(1) $\quad H_n(x) = (2x - D)^n 1, \; D \equiv d/dx, \; n \geq 0.$

When $g = \exp(-x^2/2)$, we derive the relation

(12) $\quad H_n(x) = \exp(x^2/2)(x - D)^n \exp(-x^2/2), \; n \geq 0,$

a result that is very useful in the quantum mechanical treatment of a simple harmonic oscillator (see, e.g., [2]). When $g = \exp(-x^2)$, we deduce from (3) the Rodrigues' representation (see, e.g., [5])

(13) $\quad H_n(x) = (-1)^n \exp(x^2) D^n\{\exp(-x^2)\}, \; n \geq 0.$

It is now clear that the spring of springs [i.e., (3)], the Rodrigues' representation [i.e., (13)], Arfken's formula [i.e., (1)] and (12) are all equivalent.

Relation (3) has been obtained as a natural consequence of the standard properties of the Hermite polynomials. We shall now show that (3) is a spring for developing the properties of $H_n(x)$. First we prove (9) starting from (3):

$$\begin{aligned} H_n(x) &= g^{-1}[2x - D + g^{-1}\{Dg\}]^n g \\ &= g^{-1}[2x - D + g^{-1}\{Dg\}]\{gH_{n-1}(x)\} \\ &= (2x - D)H_{n-1}(x), \; n \geq 1. \end{aligned}$$

Relation (9) plays a crucial role in establishing the results that (1) and (3) are springs of the Hermite polynomials. For example, the differential recurrence relation can be obtained from (9). If $DH_M(x) = 2MH_{M-1}(x)$ for some $M \geq 1$, then

(14) $$\begin{aligned} DH_{M+1}(x) &= D\{(2x - D)H_M(x)\} \\ &= 2H_M(x) + (2x - D)DH_M(x) \\ &= 2H_M(x) + (2x - D)\{2MH_{M-1}(x)\} \\ &= 2H_M(x) + 2MH_M(x) \\ &= 2(M + 1)H_M(x). \end{aligned}$$

By using induction, we now obtain the differential recurrence relation, (5). The three-term recurrence relation, (4), then follows from (9) and (5). The differential equation satisfied by $H_M(x)$ can be obtained from (14), since

$$2H_M(x) + (2x - D)DH_M(x) = 2(M + 1)H_M(x),$$

so that

(15) $\quad (D^2 - 2xD + 2M)H_M(x) = 0,\ M \geq 0.$

From (9), one can obtain the power series expansion (see, e.g., [5]) using induction:

$$(16) \quad H_n(x) = \sum_{s=0}^{[n/2]} \frac{(-1)^s n!(2x)^{n-2s}}{s!(n-2s)!},\ n \geq 0,$$

where $[r]$ is the greatest integer $\leq r$. Though tedious, the method is straightforward. For an alternative method of arriving at the power series expansion from (1), see also [8]. Following Simmons ([6], p. 191), we can obtain the generating function [see (2)] from the power series expansion. We show that (2) can also be derived from the pure recurrence relation as follows: (i) Assume the existence of a generating function of the form

$$(17) \quad G(x, t) = \sum_{n=0}^{\infty} H_n(x)t^n/n!$$

(ii) Differentiate $G(x, t)$ partially with respect to t and use the three-term recurrence relation *and* (6) and (7) to develop the following first-order differential equation for $G(x, t)$:

$$(18) \quad G^{-1}(\partial G/\partial t) = 2x - 2t.$$

(iii) Holding x fixed, integrate both sides of (18) with respect to t, from 0 to t, to obtain

$$(19) \quad G(x, t) = G(x, 0)\exp(2xt - t^2).$$

(iv) Since $G(x, 0) = H_0(x) = 1$, by (6), it follows that

$$(20) \quad G(x, t) = \exp(2xt - t^2).$$

Our procedure outlined above is just similar to the one used by Arfken ([2], Prob. 13.1.1, p. 717) to arrive at the generating function from the differential recurrence relation, (5), supplemented with the results

$$(21) \quad H_{2m+1}(0) = 0,\ m \geq 0,$$

$$(22) \quad H_{2m}(0) = (-1)^m(2m)!/m!,\ m \geq 0.$$

Rodrigues' representation is a simple corollary of (3) and the orthonormal property,

$$(23) \quad \int_{-\infty}^{\infty} \exp(-x^2)H_m(x)H_n(x)dx = 2^n n!\sqrt{\pi}\delta_{mn},$$

can be proved using it (see, e.g., [8]). Szegö [10] has elegantly shown that real orthogonal polynomials associated with an *even* weight function and an interval of orthogonality *symmetric* with respect to the *origin* have a definite parity. Hence,

$$(24) \quad H_n(-x) = (-1)^n H_n(x),\ n \geq 0.$$

In other words, $H_n(x)$ can contain only those powers of x that are congruent to n (mod 2). Using this result, Descartes's rule of signs, and the properties of the zeros of $H_n(x)$ (see, e.g., [5], [10]), it has been proved in [7] that $H_n(x)$ *does contain* only those and all those powers of x that are congruent to n (mod 2). Moreover, the adjacent coefficients of $H_n(x)$, $n \geq 2$, alternate in sign [7]. See also (16). Thus, starting from (3), one can obtain the differential recurrence relation, pure (i.e., without derivative) recurrence relation, differential equation, and orthonormal property satisfied by the Hermite polynomials in addition to their Rodrigues representation, power series expansion, and generating function.

3. The Relation $H_n(x) = 2^n\{\exp(-D^2/4)\}x^n$

We now prove the following interesting relation from Bell ([3], Th. 5.3, p. 159):

$$H_n(x) = 2^n\{\exp(-D^2/4)\}x^n. \tag{25}$$

Here $\exp(-D^2/4)$ is formally expanded as

$$\exp(-D^2/4) = \sum_{s=0}^{\infty}\{(-1/4)^s/s!\}D^{2s}. \tag{26}$$

Since

$$D^{2s}x^n = \begin{cases}\{n!/(n-2s)!\}x^{n-2s}, & 2s \le n,\\ 0, & 2s > n,\end{cases} \tag{27}$$

one can obtain (25) directly from the power series expansion, (16), using (26) and (27). Our proof of (25) is an alternative to that given in Bell ([3], p. 159). By retracing the steps for obtaining (25) from (16), one can show that (25) implies (16). Thus, the power series expansion and Bell's formula [i.e., (25)] are equivalent.

4. Status of the Springs

We can clearly classify the starting points into two distinct groups: (a) full/complete/self-contained springs and (b) associate (incomplete or partial) springs. To the first category belong the generating function, the Rodrigues representation, the power series expansion, relations (1), (3), and (25), and the orthonormal property. These springs specify the Hermite polynomials completely. The differential equation, the pure and differential recurrence relations, the orthogonal property, and (9) belong to the second category because they require supplementary conditions to specify the Hermite polynomials fully. The constant term of any $H_n(x)$, $n \ge 1$, cannot be found from the differential recurrence relation, (5), since the operator D simply swallows it. In the case of the orthogonal property, we require the value of the right-hand side of (23) when $m = n$, for all $n \ge 0$ (the square root of the reciprocal of this quantity is the so-called normalization constant), and to make (9) a complete spring we require the result $H_0(x) = 1$.

An outline of the development of the various properties from the springs is shown schematically in Figure 1. (Of course, not all the paths are shown.) Certain properties can be more easily obtained from a given spring, while it may be tedious to derive another property from the same spring. For example, in view of (26), we have

$$[D\exp(-D^2/4)]f(x) \equiv \{\exp(-D^2/4)\}(Df),$$

where $f(x)$ is any differentiable function of x. Hence, from (25) and (26), we have

$$\begin{aligned}DH_n(x) &= D[2^n\{\exp(-D^2/4)\}x^n]\\ &= 2^n\{\exp(-D^2/4)\}(Dx^n)\\ &= 2n[2^{n-1}\{\exp(-D^2/4)\}x^{n-1}]\\ &= 2nH_{n-1}(x),\ n \ge 1.\end{aligned}$$

Probably this is the simplest proof of the differential recurrence relation. The method of induction plays an elegant role in developing certain properties from a given starting point. Some properties can be independently obtained from a given spring without going either via the generating function or via the Rodrigues representation.

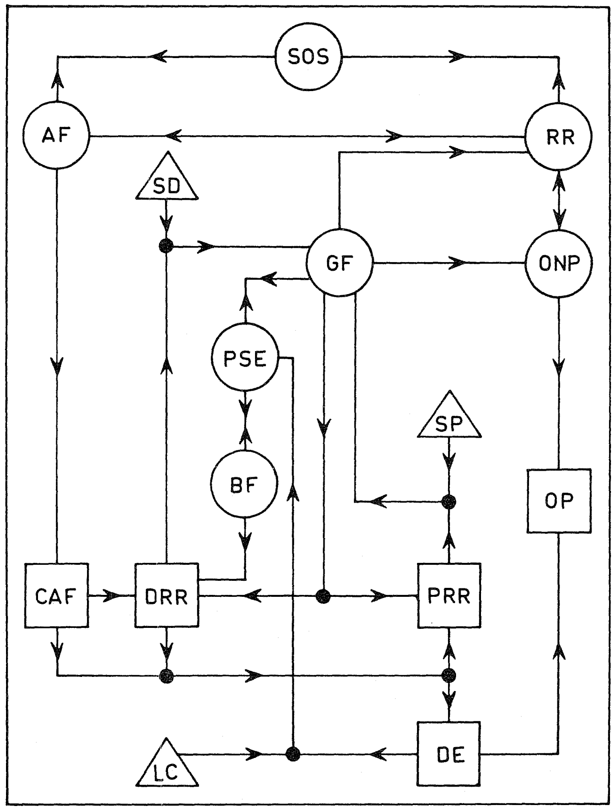

FIGURE 1

Schematic diagram showing the development of the various properties of the Hermite polynomials. Full springs are shown inside the circles. Squares enclose the associate starting points. Triangles stand for the supplementary conditions necessary to make the incomplete springs complete ones. We have not given the complete paths to arrive at all the properties, starting from a given spring. The following abbreviations have been used: (a) AF: Arfken's formula, (1) of text. (b) BF: Bell's formula, (25) of text. (c) CAF: Corollary to Arfken's formula, (9) of text. (d) DE: Differential equation. (e) DRR: Differential recurrence relation. (f) GF: Generating function. (g) LC: Leading coefficient of each and every $H_n(x)$, $n \geq 0$ $(= 2^n)$; supplement to the differential equation. (h) ONP: Orthonormal property. (i) OP: Orthogonal property. (A knowledge of the leading coefficient or the normalization constant for every $H_n(x)$ makes it a complete spring.) (j) PRR: Pure (three-term) recurrence relation. (k) PSE: Power series expansion. (l) RR: Rodrigues' representation. (m) SD: Supplement to the differential recurrence relation, (21) and (22) of text. (n) SOS: Spring of springs, (3) of text. (o) SP: Supplement to the pure recurrence relation, (6) and (7) of text.

5. Conclusions

Any relation or a set of relations that can specify all the Hermite polynomials completely should be a full starting point. One can level criticisms against any spring. For Simmons ([6], p. 189), the generating function method is totally unmotivated, though it has the advantage of efficiency for deducing the properties of the Hermite polynomials. While he prefers to develop the properties from the differential equation, Andrews ([1], p. vii) introduces the classical orthogonal polynomials by the generating function method and Rainville [5] revels in the generating function approach. Relation (1) is simple and handy, but may have the obvious weakness of being completely unmotivated.

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