TILING THE k^{th} POWER OF A POWER SERIES

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In this paper we consider the problem of dividing a rectangle into non-overlapping squares and rectangles using recurring-sequence tiling. The results obtained herein are illustrated with appropriate figures. These results, with the exception of basic introductory material, are believed to be new. There seem to be no such results in the literature.

Among the many generating functions possible, we use the following:

(1) $$G(x)^{-k} = 1/(1 - x - x^2 - \cdots - x\ )$$
(where $m = 2, 3, 4, \ldots$, and $k = 1, 2, 3, \ldots$).

Note that we can write $G(x)^{-k}$ as a power series in x in the form

(2) $$G(x)^{-k} = F_{m,k}(0) + F_{m,k}(1)x + F_{m,k}(2)x^2 + \cdots + F_{m,k}(n)x^n + \cdots$$
(where $F_{m,k}(0) = 1$, for all m and k).

We develop a general construction method for performing the tiling using $k \geq 2$. Our work is an extension of the tiling done in [7] for $k = 1$.

Using (2),

(3) $$G(x)^{-(k-1)} = (1 - x - x^2 - \cdots - x^m)(F_{m,k}(0) + F_{m,k}(1)x + F_{m,k}(2)x^2 + \cdots + F_{m,k}(n)x^n + \cdots).$$

Now, combining coefficients in Equation (3) leads to

(4) $$F_{m,k}(n) = F_{m,k}(n-1) + F_{m,k}(n-2) + \cdots + F_{m,k}(n-m) + F_{m,k-1}(n).$$

The last term of Equation (4) is important. To preserve the geometry of the method of tiling we have used in this paper, it is necessary that

(5) $$F_{m,k-1}(n) < F_{m,k}(n-m),$$

where $F_{m,k-1}(n)$ is the value of the initial tile placed in the construction.

First let $m = 2$; we shall find the sizes of tiles corresponding to values of $F_{2,k}(n)$ for various values of k. In Table 1, we have outlined the value of the initial tiles generated by the necessary condition that

$$F_{m,k-1}(n) < F_{m,k}(n-m)$$

as discussed above. For example, note that $420 < 474$, $23109 < 25088$, etc.

The values in Table 1 can be used in an example tiling construction for $m = 2$ and $k = 2$, as shown in Figure 2. Note that each shape is a square.

For higher-order constructions, the difficulty lies in choosing the initial tile. We now concern ourselves with finding the value of that required initial tile. A repeated use of Equation (4) above, in example cases of $m = 2$ and $m = 3$, is summarized in Tables 3 and 4.

TABLE 1. Coefficients $F_k(n)$ when $m = 2$

n	k = 1	k = 2	k = 3	k = 4	k = 5
0	1	1	1	1	1
1	1	2	3	4	5
2	2	5	9	14	20
3	3	10	22	40	65
4	5	20	51	105	190
5	8	38	111	256	511
6	13	71	233	594	1295
7	21	130	474	1324	3130
8	34	235	942	2860	7285
9	55	$\boxed{420}$	1836	6020	16435
10	89	744	3522	12402	36122
11	144	1308	6666	25088	77645
12	233	2285	12473	49963	163730
13	377	3970	$\boxed{23109}$	98160	339535
14	610	6865	42447	190570	693835
15	987	11822	77378	366108	1399478
16	1597	20284	140109	696787	2790100
17	2584	34690	252177	$\boxed{1315072}$	5504650
18	4181	59155	451441	2463300	10758050
19	6765	100610	804228	4582600	20845300
20	10946	170711	1426380	8472280	40075630

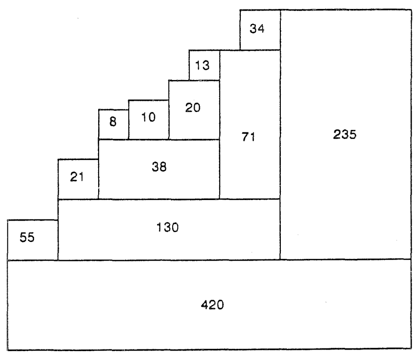

FIGURE 2

TABLE 3. Calculating the Initial Tile for $m = 2$

k = 2	$F_{2,1}(5) < F_{2,2}(3)$
k = 3	$F_{2,2}(9) < F_{2,3}(7)$
k = 4	$F_{2,3}(13) < F_{2,4}(11)$
k = 5	$F_{2,4}(17) < F_{2,5}(15)$
k = 6	$F_{2,5}(21) < F_{2,6}(19)$

TABLE 4. Calculating the Initial Tile for $m = 3$

k = 2	$F_{3,1}(12) < F_{3,2}(9)$
k = 3	$F_{3,2}(23) < F_{3,3}(20)$
k = 4	$F_{3,3}(34) < F_{3,4}(31)$
k = 5	$F_{3,4}(45) < F_{3,5}(42)$
k = 6	$F_{3,5}(56) < F_{3,6}(53)$

Surveying the values above suggests the following relationships. Define $Q_{m,k}$ to be the smallest number such that

$$F_{m,k-1}(Q_{m,k} + m) < F_{m,k}(Q_{m,k}),$$

and let $Q_m = Q_{m,1}$. Let $Q_0 = Q_1 = 1$. Then from Table 3, where $m = 2$, we observe that

(6) $\quad Q_2 = 3Q_1 - 2Q_0 + 2.$

Further, we observe from Table 4, where $m = 3$, that

(7) $\quad Q_3 = 3Q_2 - 2Q_1 + 2.$

We then generalize that

(8) $\quad Q_m = 3Q_{m-1} - 2Q_{m-2} + 2.$

By elementary means we find that Equation (8) can be stated as

(9) $\quad Q_m = 2^{m+1} - 2m - 1.$

Then $F_{m,k}(Z_m)$ is the initial tile I, where $I = Z_m = 2^{m+1} - m - 1$, for $m \geq 2$.

An examination of Tables 3 and 4 will show that a pattern emerges as k changes and one looks for a value of n which will result in the next initial tile value. As k changes by one, the value of n changes by a constant amount. That constant is equal to

(10) $\quad P_m = 2^{m+1} - m - 2.$

It can be shown inductively, step by step, that the values of the initial tiles are

(11) $\quad F_{m,k-1}[Z_m + (k-2)P_m] < F_{m,k}[Q_m + (k-2)P_m]$, where $m \geq 2$ and $k \geq 1$.

Then, for example, the next tile values in Table 3, using (11), are seen to be

$$F_{2,6}(25) < F_{2,7}(23).$$

Finally, we now show the general case of placing tiles in the construction. We place the sets of squares in the order Set 1, Set 2, etc., where the Sets are defined as

(12) Set 1: $F_{m,k}(n) = F_{m,k}(n-1) + F_{m,k}(n-2) + \cdots + F_{m,k}(n-m) + F_{m,k-1}(n)$

(13) Set 2: $F_{m,k}(n+m-1) = F_{m,k}(n+m-2) + F_{m,k}(n+m-3) + \cdots + F_{m,k}(n-1) + F_{m,k-1}(n+m-1)$

(14) Set 3: $F_{m,k}(n+2m-2) = F_{m,k}(n+2m-3) + F_{m,k}(n+2m-4) + \cdots + F_{m,k}(n+m-2) + F_{m,k-1}(n+2m-2)$

...

(15) Set $j+1$: $F_{m,k}(n+j(m-1))$
$= F_{m,k}(n-1+j(m-1)) + F_{m,k}(n-2+j(m-1)) + \cdots + F_{m,k}(n+m+j(m-1)) + F_{m,k-1}(n+j(m-1))$,

where $j = 0, 1, 2, \ldots.$

Using this method of set placement, the final general construction will look like Figure 5 below. Note that all shapes are squares, and that as of yet we have not fully tiled the rectangle; there are still rectangular gaps in the construction.

We now proceed to find the filler rectangles used to fill in the gaps left after placing the square tiles. We note first that the general coefficients $F_{m,k}(n)$ may be listed as follows, where $m \geq 2$, $k \geq 1$, and $n \geq 0$.

(16) $F_{m,k}(0)$
$F_{m,k}(1)$
$F_{m,k}(2)$
...
$F_{m,k}(I)$
...

where $F_{m,k}(I) = F_{m,k}(I-1) + F_{m,k}(I-2) + \cdots + F_{m,k}(I-m+1) + F_{m,k}(I-m) + F_{m,k-1}(I)$; $I = Z_m + (k-1)P_m$; $F_{m,0}(I) = 0$, $F_{m,k}(0) = 1$, and $F_{m,k}(1) = k$.

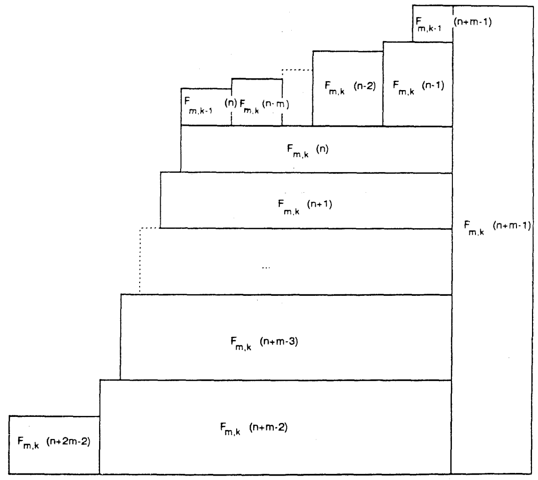

FIGURE 5

Note that I is the initial tile value, and it is evident that the value of I depends on the values of k and m.

First we examine the case of $m = 2$ and $k = 2$ [in other words, $1/(1 - x - x^2)^2$]. Figure 2 may now be redrawn using function notation rather than actual numbers, and showing the gap rectangles. We use the notation H and V to denote filler rectangles that appear to be oriented horizontally and vertically, respectively.

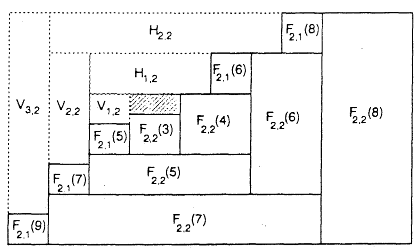

FIGURE 6

One can see the following rectangle sizes from Figure 6.

(17)
$$\begin{aligned}
&H_{1,k}[F_{2,2}(5) - F_{2,1}(6),\ F_{2,1}(6)]\\
&H_{2,k}[F_{2,2}(7) - F_{2,1}(8),\ F_{2,1}(8)]\\
&\ldots\\
&H_{n,k}[F_{2,2}(2n+5) - F_{2,1}(2n+6),\ F_{2,1}(2n+6)]\\
&V_{1,k}[F_{2,1}(5),\ F_{2,2}(4) - F_{2,1}(5)]\\
&V_{2,k}[F_{2,1}(7),\ F_{2,2}(6) - F_{2,1}(7)]\\
&\ldots\\
&V_{n,k}[F_{2,1}(2n+3),\ F_{2,2}(2n+2) - F_{2,1}(2n+5)],
\end{aligned}$$

where $n \geq 0$, $m = 2$, and $k = 2$.

We can generalize this idea for any k. To use the I notation, Figure 6 may be redrawn as follows:

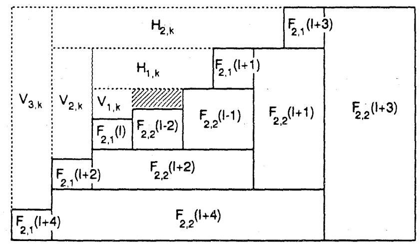

FIGURE 7

Furthermore, (17) can be rewritten as

(18)
$$H_{1,k}[F_{2,k}(I) - F_{2,k}(I+1),\ F_{2,k-1}(I+1)]$$
$$H_{2,k}[F_{2,k}(I+2) - F_{2,k-1}(I+3),\ F_{2,k-1}(I+3)]$$
$$\cdots$$
$$H_{n,k}[F_{2,k}(2n+I) - F_{2,k-1}(2n+I+1),\ F_{2,k-1}(2n+I+1)$$
$$V_{1,k}[F_{2,k-1}(I),\ F_{2,k}(I-1) - F_{2,k-1}(I)]$$
$$V_{2,k}[F_{2,k-1}(I+2),\ F_{2,k}(I+1) - F_{2,k-1}(I+2)]$$
$$\cdots$$
$$V_{n,k}[F_{2,k-1}(2n+I-2),\ F_{2,k}(2n+I-3) - F_{2,k-1}(2n+I-2),$$

where $n \geq 0$, $m = 2$, and $k \geq 2$.

We now show an example for $m > 2$, in particular for $m = 3$, $k = 2$. In this case, the tiling construction begins with the framework shown in Figure 8, using (4) for recursion and Table 4 to determine the initial tile.

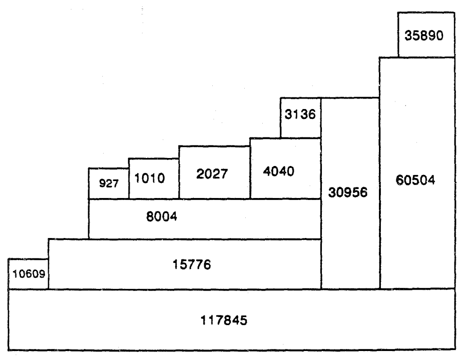

FIGURE 8

The filler rectangles are formed in a similar manner as the case of $m = 2$, except an alternating pattern containing two different constructions is formed. The complete construction for $k = 2$, $m = 3$ is shown in Figure 9. This same pattern is followed for any k, where only the value of the arguments of the function F is changed as influenced by the change in the initial tile.

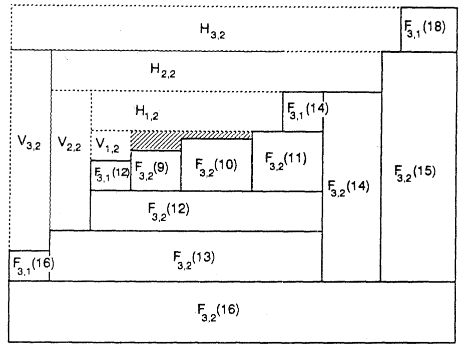

FIGURE 9

One can see the following rectangle sizes from Figure 9:

$$
\begin{aligned}
(19)\qquad & H_{1,3}[F_{3,2}(12) - F_{3,1}(14),\ F_{3,1}(14)]\\
& H_{2,2}[F_{3,2}(13) + F_{3,2}(14),\ F_{3,2}(15) - F_{3,2}(14)]\\
& \ldots\\
& V_{1,2}[F_{3,1}(12),\ F_{3,2}(11) - F_{3,1}(12)]\\
& V_{2,2}[F_{3,2}(13) - F_{3,2}(12),\ F_{3,2}(14) - F_{3,2}(13)]\\
& \ldots .
\end{aligned}
$$

The pattern of the pair of equations in both the horizontal and vertical rectangles is repeated. These formulas would also be valid for any value of k, not just $k = 2$.

We can now generalize this idea for any m. The pattern is repeated every $m - 1$ rectangles. Therefore, the difference in the argument where this pattern repeats is $2m - 2$. Figure 10 shows the general construction.

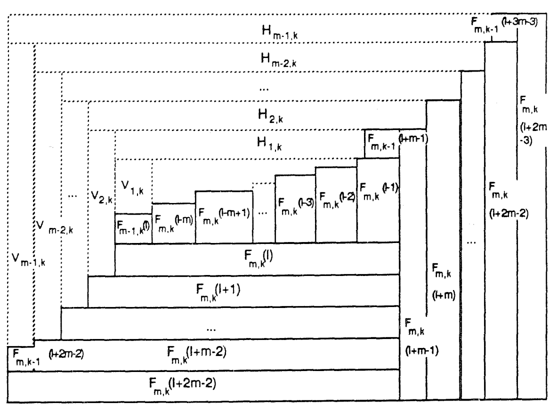

FIGURE 10

The general formulas for the horizontal and vertical rectangles are:

$$
\begin{aligned}
(20)\qquad & H_{1,k}[F_{m,k}(I) - F_{m,k-1}(I + m - 1),\ F_{m,k-1}(I + m - 1)]\\
& H_{2,k}[F_{m,k}(I + 1) + F_{m,k}(I + m - 1),\ F_{m,k}(I + m) - F_{m,k}(I + m - 1)]\\
& H_{3,k}[F_{m,k}(I + 2) + F_{m,k}(I + m - 1) + F_{m,k}(I + m),\\
& \qquad F_{m,k}(I + m + 1) - F_{m,k}(L + m)]\\
& \ldots\\
& H_{m-1,k}[F_{m,k}(I + m - 2) + F_{m,k}(I + m - 1) + F_{m,k}(I + m)\\
& \qquad + \cdots + F_{m,k}(I + 2m - 4),\ F_{m,k}(I + 2m - 3) - F_{m,k}(I + 2m - 4)]
\end{aligned}
$$

$$\begin{aligned}
&\ldots\\
&V_{1,k}[F_{m,k-1}(I),\ F_{m,k}(I-1) - F_{m,k-1}(I)]\\
&V_{2,k}[F_{m,k}(I+1) - F_{m,k}(I),\ F_{m,k}(I+m-1) - F_{m,k}(I+m-2)\\
&\quad - \cdots - F_{m,k}(m+2) - F_{m,k}(m+1)]\\
&V_{3,k}[F_{m,k}(I+2) - F_{m,k}(I+1),\ F_{m,k}(I+m) - F_{m,k}(I+m-2)\\
&\quad - F_{m,k}(I+m-3) - \cdots - F_{m,k}(I+2)]\\
&\ldots\\
&V_{m-2,k}[F_{m,k}(I+m-3) - F_{m,k}(I+m-4),\\
&\qquad F_{m,k}(I+2m-5) - F_{m,k}(I+m-2) - F_{m,k}(I+m-3)]\\
&V_{m-1,k}[F_{m,k}(I+m-2) - F_{m,k}(I+m-3),\\
&\qquad F_{m,k}(I+2m-4) - F_{m,k}(I+m-2)].
\end{aligned}$$

Then the pattern of $m - 1$ rectangles repeats with the argument (I) incremented by $2m - 2$.

This construction, shown in Figure 10, generalizes the recurring-sequence tiling for any k and m using the extended Fibonacci numbers generated with (4). The initial tile is selected by the criterion of (5) and the horizontal and vertical filler rectangles have the dimensions described in (20).

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