A TWO-VARIABLE LAGRANGE-TYPE INVERSION FORMULA WITH APPLICATIONS TO EXPANSION AND CONVOLUTION IDENTITIES

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1. Introduction

Using Lagrange inversion, one obtains the formal power series (fps) expansions (cf. Riordan [12], Sec. 4.5)

$$(1.1)\qquad \exp bz = \sum_{k=0}^{\infty} \frac{b(ak+b)^{k-1}}{k!} w^k,$$

$$(1.2)\qquad \frac{\exp bz}{1-az} = \sum_{k=0}^{\infty} \frac{(ak+b)^k}{k!} w^k,$$

where $w = z \cdot \exp(-az)$, and

$$(1.3)\qquad (1+z)^b = \sum_{k=0}^{\infty} \frac{b}{ak+b} \binom{ak+b}{k} v^k,$$

$$(1.4)\qquad \frac{(1+z)^b}{1-\dfrac{az}{1+z}} = \sum_{k=0}^{\infty} \binom{ak+b}{k} v^k,$$

where $v = z/(1+z)^a$. With the help of these identities, Gould [6-8] obtained many convolution identities. Higher-dimensional extensions of (1.2) and (1.4) were studied and proved by Carlitz [1, 2] using MacMahon's Master Theorem. Finally, Carlitz's identities were embedded into a general theory by Joni [9]. The key for her results, again, is Lagrange inversion (this time the multivariable Lagrange-Good inversion formula, cf. Joni [10]).

In [5] Cohen & Hudson discovered two-variable generalizations of (1.1) and (1.2) that are different in nature from the corresponding results of Carlitz, and studied related convolution identities. Their proofs are based on a specific operator method also used in Cohen's papers [3] and [4]. Thus, the question remained open as to whether there might be a Lagrange-type inversion formula providing the background for Cohen & Hudson's results and yielding two-variable extensions of (1.3) and (1.4), in addition. This formula will be given in Section 3 (Theorem 1, Corollary 2) of this paper. Subsequently, we are able to derive all of Cohen & Hudson's results and, moreover, to give the "factorial" analogues that correspond to (1.3) and (1.4). This will be done in Section 4. For the purpose of illustration, we list some identities in the next section.

2. Some Expansion and Convolution Identities

To write our identities, it is convenient to adopt the usual multidimensional notations. Let $\underset{\sim}{k} = (k_1, k_2)$, $\underset{\sim}{n} = (n_1, n_2) \in \mathbb{Z}^2$ (pairs of integers) and $\underset{\sim}{z} = (z_1, z_2)$ be a pair of indeterminates, then we define $\underset{\sim}{k}! = k_1!k_2!$, $\underset{\sim}{n} \ge \underset{\sim}{k}$ if and only if $n_1 \ge k_1$ and $n_2 \ge k_2$, $\underset{\sim}{n} - \underset{\sim}{k} = (n_1 - k_1, n_2 - k_2)$, $\underset{\sim}{0} = (0, 0)$,

$$\underset{\sim}{z}^{\underset{\sim}{k}} = z_1^{k_1} z_2^{k_2} \quad\text{and}\quad \binom{\underset{\sim}{z}}{\underset{\sim}{k}} = \binom{z_1}{k_1}\binom{z_2}{k_2}.$$

Throughout this paper, for $\underset{\sim}{k} \in \mathbb{Z}^2$, λ_i, μ_i, α_i, $\beta_i \in \mathbb{C}$ (complex numbers), $i = 1, 2$, we shall write

$$R_1(\underset{\sim}{k}) = \frac{(\lambda_1 + \mu_1 k_1)(\alpha_2 + \beta_2 k_2)}{\lambda_2 + \mu_2 k_2}, \quad R_2(\underset{\sim}{k}) = \frac{(\lambda_2 + \mu_2 k_2)(\alpha_1 + \beta_1 k_1)}{\lambda_1 + \mu_1 k_1},$$

and

$$r_1(\underset{\sim}{k}) = \frac{(\lambda_1 + \mu_1 k_1)}{(\lambda_2 + \mu_2 k_2)}, \qquad r_2(\underset{\sim}{k}) = \frac{(\lambda_2 + \mu_2 k_2)}{(\lambda_1 + \mu_1 k_1)},$$

for short. Note that $r_i(\underset{\sim}{k})$ is equal to $R_i(\underset{\sim}{k})$ with $\beta_i = 0$ and $\alpha_i = 1$.

The first of Cohen & Hudson's identities, (1.3) in [5], is equivalent to

$$(2.1) \qquad \frac{1 - \dfrac{\mu_1 \mu_2}{\lambda_1 \lambda_2} z_1 z_2}{\left(1 - \dfrac{\mu_1}{\lambda_2} z_1\right)\left(1 - \dfrac{\mu_2}{\lambda_1} z_2\right)} = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{r_1(\underset{\sim}{k})^{k_1}}{k_1!} \frac{r_2(\underset{\sim}{k})^{k_2}}{k_2!} \underset{\sim}{z}^{\underset{\sim}{k}} \exp(-r_1(\underset{\sim}{k}) z_1 - r_2(\underset{\sim}{k}) z_2).$$

For $z_2 = 0$, (2.1) reduces to (1.2). The factorial analogue of (2.1) we prove is

$$(2.2) \qquad \frac{1 - \dfrac{\mu_1 \mu_2}{\lambda_1 \lambda_2} \dfrac{z_1 z_2}{(1 + z_1)(1 + z_2)}}{\left(1 - \dfrac{\mu_1}{\lambda_2} \dfrac{z_1}{1 + z_1}\right)\left(1 - \dfrac{\mu_2}{\lambda_1} \dfrac{z_2}{1 + z_2}\right)}$$
$$= \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \binom{r_1(\underset{\sim}{k})}{k_1} \binom{r_2(\underset{\sim}{k})}{k_2} \underset{\sim}{z}^{\underset{\sim}{k}} (1 + z_1)^{-r_1(\underset{\sim}{k})} (1 + z_2)^{-r_2(\underset{\sim}{k})}.$$

Equation (2.2) reduces to (1.4) for $z_2 = 0$. The "mixed" expansion,

$$(2.3) \qquad \frac{1 - \dfrac{\mu_1 \mu_2}{\lambda_1 \lambda_2} z_1 \dfrac{z_2}{1 + z_2}}{\left(1 - \dfrac{\mu_1}{\lambda_2} z_1\right)\left(1 - \dfrac{\mu_2}{\lambda_1} \dfrac{z_2}{1 + z_2}\right)} + \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{r_1(\underset{\sim}{k})^{k_1}}{k_1!} \binom{r_2(\underset{\sim}{k})}{k_2} \underset{\sim}{z}^{\underset{\sim}{k}} \exp(-r_1(\underset{\sim}{k}))(1 + z_2)^{-r_2(\underset{\sim}{k})},$$

is a two-variable generalization of (1.2) and (1.4) at the same time. This is seen by setting $z_2 = 0$ or $z_1 = 0$, respectively.

The second expansion of Cohen & Hudson, (1.5) in [5], is equivalent to

$$(2.4) \qquad \frac{1}{\lambda_1 - \mu_2 \alpha_1 z_2} = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{1}{\lambda_1 + \mu_1 k_1} \frac{r_1(\underset{\sim}{k})^{k_1} R_2(\underset{\sim}{k})^{k_2}}{\underset{\sim}{k}!} \underset{\sim}{z}^{\underset{\sim}{k}} \exp(-r_1(\underset{\sim}{k}) z_1 - R_2(\underset{\sim}{k}) z_2).$$

Setting $z_2 = 0$ in (2.4) gives (1.1); setting $z_1 = 0$ gives (1.2). The factorial analogue of (2.4) is presented here:

$$(2.5) \qquad \frac{1}{\lambda_1 - \mu_2 \alpha_1 \dfrac{z_2}{1 + z_2}} = \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{1}{\lambda_1 + \mu_1 k_1} \binom{r_1(\underset{\sim}{k})}{k_1} \binom{R_2(\underset{\sim}{k})}{k_2} \underset{\sim}{z}^{\underset{\sim}{k}} (1 + z_1)^{-r_1(\underset{\sim}{k})} (1 + z_2)^{-R_2(\underset{\sim}{k})}.$$

This is a generalization of (1.3) and (1.4) at the same time. Similarly, a two-dimensional generalization of (1.1) and (1.4) is

$$(2.6) \qquad \frac{1}{\lambda_1 - \mu_2 \alpha_1 \dfrac{z_2}{1 + z_2}}$$
$$= \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} \frac{1}{\lambda_1 + \mu_1 k_1} \frac{r_1(\underset{\sim}{k})^{k_1}}{k_1!} \binom{R_2(\underset{\sim}{k})}{k_2} \underset{\sim}{z}^{\underset{\sim}{k}} \exp(-r_1(\underset{\sim}{k}) z_1)(1 + z_2)^{-R_2(\underset{\sim}{k})},$$

and a generalization of (1.2) and (1.3) is

$$\frac{1}{\lambda_1 - \mu_2\alpha_1 z_2} = \sum_{\mathbf{k} \geq \mathbf{0}} \frac{1}{\lambda_1 + \mu_1 k_1}\binom{r_1(\mathbf{k})}{k_1}\frac{R_2(\mathbf{k})^{k_2}}{k_2!}\mathbf{z}^{\mathbf{k}}(1 + z_1)^{-r_1(\mathbf{k})}\exp(-R_2(\mathbf{k})z_2). \tag{2.7}$$

We give another expansion, which Cohen & Hudson missed:

$$1 = \sum_{\mathbf{k} \geq \mathbf{0}} \frac{\lambda_1\lambda_2 - k_1k_2\dfrac{(\alpha_1\mu_1 - \beta_1\lambda_1)(\alpha_2\mu_2 - \beta_2\lambda_2)}{(\alpha_1 + \beta_1 k_1)(\alpha_2 + \beta_2 k_2)}}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)} \frac{R_1(\mathbf{k})^{k_1} R_2(\mathbf{k})^{k_2}}{\mathbf{k}!} \cdot \mathbf{z}^{\mathbf{k}} \exp(-R_1(\mathbf{k})z_1 - R_2(\mathbf{k})z_2). \tag{2.8}$$

This is a two-variable generalization of (1.1). The corresponding generalization of (1.3) reads

$$1 = \sum_{\mathbf{k} \geq \mathbf{0}} \frac{\lambda_1\lambda_2 - k_1k_2\dfrac{(\alpha_1\mu_1 - \beta_1\lambda_1)(\alpha_2\mu_2 - \beta_2\lambda_2)}{(\alpha_1 + \beta_1 k_1)(\alpha_2 + \beta_2 k_2)}}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)} \binom{R_1(\mathbf{k})}{k_1}\binom{R_2(\mathbf{k})}{k_2} \cdot \mathbf{z}^{\mathbf{k}}(1 + z_1)^{-R_1(\mathbf{k})}(1 + z_2)^{-R_2(\mathbf{k})}, \tag{2.9}$$

that of (1.1) and (1.3),

$$1 = \sum_{\mathbf{k} \geq \mathbf{0}} \frac{\lambda_1\lambda_2 - k_1k_2\dfrac{(\alpha_1\mu_1 - \beta_1\lambda_1)(\alpha_2\mu_2 - \beta_2\lambda_2)}{(\alpha_1 + \beta_1 k_1)(\alpha_2 + \beta_2 k_2)}}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)} \frac{R_1(\mathbf{k})^{k_1}}{k_1!}\binom{R_2(\mathbf{k})}{k_2} \cdot \mathbf{z}^{\mathbf{k}} \exp(-R_1(\mathbf{k})z_1)(1 + z_2)^{-R_2(\mathbf{k})}. \tag{2.10}$$

Identities (2.1)-(2.10) will be proved in Section 4 by establishing three types of general expansions that underly (2.1)-(2.3), (2.4)-(2.7), and (2.8)-(2.10), respectively.

From each of the expansions (2.1)-(2.10), we may derive a convolution identity which generalizes Jensen's convolution or the Abel-type Jensen-Gould convolution identity, respectively (cf. [6], [7]). We shall give two examples; the remaining identities are obtained similarly.

Multiplying both sides of (2.1) with $\exp(s_1z_1 + s_2z_2)$ and comparing coefficients of $\mathbf{z}^{\mathbf{n}}$, we obtain (1.4) of [5].

$$\frac{s_2^{n_2}}{n_2!}\sum_{j_1=0}^{n_1}\left(\frac{\mu_1}{\lambda_2}\right)^{j_1}\frac{s_1^{n_1-j_1}}{(n_1 - j_1)!} + \frac{s_1^{n_1}}{n_1!}\sum_{j_2=0}^{n_2}\left(\frac{\mu_2}{\lambda_1}\right)^{j_2}\frac{s_2^{n_2-j_2}}{(n_2 - j_2)!} - \frac{\mathbf{s}^{\mathbf{n}}}{\mathbf{n}!} = \sum_{\mathbf{k} \geq \mathbf{0}} \frac{r_1(\mathbf{k})^{k_1} r_2(\mathbf{k})^{k_2}}{\mathbf{k}!} \frac{(-r_1(\mathbf{k}) + s_1)^{n_1-k_1} \cdot (-r_2(\mathbf{k}) + s_2)^{n_2-k_2}}{(\mathbf{n} - \mathbf{k})!}. \tag{2.11}$$

The factorial analogue of (2.11), deduced by multiplying both sides of (2.2) with $(1 + z_1)^{s_1}(1 + z_2)^{s_2}$ and comparing coefficients of $\mathbf{z}^{\mathbf{n}}$ reads

$$\binom{s_2}{n_2}\sum_{j_1=0}^{n_1}\left(\frac{\mu_1}{\lambda_2}\right)^{j_1}\binom{s_1 - j_1}{n_1 - j_1} + \binom{s_1}{n_1}\sum_{j_2=0}^{n_2}\left(\frac{\mu_2}{\lambda_1}\right)^{j_2}\binom{s_2 - j_2}{n_2 - j_2} - \binom{\mathbf{s}}{\mathbf{n}} = \sum_{\mathbf{k} \geq \mathbf{0}}\binom{r_1(\mathbf{k})}{k_1}\binom{r_2(\mathbf{k})}{k_2}\binom{-r_1(\mathbf{k}) + s_1}{n_1 - k_1}\binom{-r_2(\mathbf{k}) + s_2}{n_2 - k_2}. \tag{2.12}$$

3. Lagrange Inversion

Let

$$\phi(\mathbf{z}) = (\phi_1(z_1, z_2), \phi_2(z_1, z_2))$$

be some pair of fps in z_1 and z_2 with $\phi_i(0, 0) \neq 0$, $i = 1, 2$. Let

$$f(\mathbf{z}) = (z_1/\phi_1(\mathbf{z}),\ z_2/\phi_2(\mathbf{z})).$$

The (two-variable) Lagrange-Good formula solves the problem of expanding a formal Laurent series $g(\mathbf{z})$ of the form

$$(3.1)\qquad g(\mathbf{z}) = \sum_{\mathbf{j} \geq \mathbf{m}} g_{\mathbf{j}} \mathbf{z}^{\mathbf{j}},$$

for some $\mathbf{m} \in \mathbf{Z}^2$, in terms of powers of $f(\mathbf{z})$; namely, if

$$g(\mathbf{z}) = \sum_{\mathbf{k} \in \mathbf{Z}^2} c_{\mathbf{k}} f^{\mathbf{k}}(\mathbf{z}),$$

then

$$c_{\mathbf{k}} = \langle \mathbf{z}^{\mathbf{0}} \rangle g(\mathbf{z}) \Delta(\mathbf{z}) f^{-\mathbf{k}}(\mathbf{z}),$$

where $\langle \mathbf{z}^{\mathbf{0}} \rangle a(\mathbf{z})$ denotes the coefficient of $\mathbf{z}^{\mathbf{0}}$ in $a(\mathbf{z})$, and

$$\Delta(\mathbf{z}) = \frac{\partial f}{\partial \mathbf{z}}(\mathbf{z}) \phi_1(\mathbf{z}) \phi_2(\mathbf{z}) \text{ with } \frac{\partial f}{\partial \mathbf{z}}(\mathbf{z}) \text{ the Jacobian of } f(\mathbf{z}).$$

For formal Laurent series of the form (3.1), we shall use the abbreviation fLs.

The general two-dimensional Lagrange inversion problem can be formulated as follows: Let $F = (f_{\mathbf{k}}(\mathbf{z}))_{\mathbf{k} \in \mathbf{Z}^2}$ be a "diagonal sequence," i.e., $f_{\mathbf{k}}(\mathbf{z})$ is of the form

$$f_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{n} \geq \mathbf{k}} f_{\mathbf{nk}} \mathbf{z}^{\mathbf{k}}.$$

Then, for a given sequence F, one tries to find some sequence $\tilde{F} = (\tilde{f}_{\mathbf{k}}(\mathbf{z}))_{\mathbf{k} \in \mathbf{Z}^2}$ such that expanding an arbitrary fLs $g(\mathbf{z})$ in terms of F,

$$(3.2a)\qquad g(\mathbf{z}) = \sum_{\mathbf{k} \in \mathbf{Z}^2} c_{\mathbf{k}} f_{\mathbf{k}}(\mathbf{z}),$$

the coefficients $c_{\mathbf{k}}$ are given by

$$(3.2b)\qquad c_{\mathbf{k}} = \langle \mathbf{z}^{\mathbf{0}} \rangle g(\mathbf{z}) \cdot \tilde{f}_{\mathbf{k}}(\mathbf{z}).$$

Obviously, the sequence $\tilde{F}$ is uniquely determined by

$$(3.3)\qquad \langle \mathbf{z}^{\mathbf{0}} \rangle f_{\mathbf{k}}(\mathbf{z}) \cdot \tilde{f}_{\mathbf{n}}(\mathbf{z}) = \delta_{\mathbf{nk}},$$

where $\delta_{\mathbf{nk}}$ is the Kronecker delta. In this paper, we shall solve this Lagrange inversion problem for

$$(3.4)\qquad f_{\mathbf{k}}(\mathbf{z}) = \mathbf{z}^{\mathbf{k}} f_1(z_1)^{R_1(\mathbf{k})} f_2(z_2)^{R_2(\mathbf{k})},$$

where $f_1(t)$, $f_2(t)$ are fps in the single variable t with $f_i(0) \neq 0$, $i = 1, 2$. Evidently, $F = (f_{\mathbf{k}}(\mathbf{z}))_{\mathbf{k} \in \mathbf{Z}^2}$ is a diagonal sequence.

Theorem 1: Let $F = (f_{\mathbf{k}}(\mathbf{z}))_{\mathbf{k} \in \mathbf{Z}^2}$ be as defined in (3.4). The sequence

$$\tilde{F} = (\tilde{f}_{\mathbf{k}}(\mathbf{z}))_{\mathbf{k} \in \mathbf{Z}^2},$$

uniquely determined by (3.3), is given by

$$(3.5)\qquad \tilde{f}_{\mathbf{k}}(\mathbf{z}) = \frac{\mu_1 \mu_2}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)} W f_{\mathbf{k}}(\mathbf{z})^{-1},$$

where W is the operator

$$(3.6)\qquad W = \det \begin{vmatrix} -z_1 D_1 + \dfrac{\lambda_1}{\mu_1} & \left(\alpha_2 - \dfrac{\beta_2}{\mu_2}\lambda_2\right) F_1(z_1) \\ \left(\alpha_1 - \dfrac{\beta_1}{\mu_1}\lambda_1\right) F_2(z_2) & -z_2 D_2 + \dfrac{\lambda_2}{\mu_2} \end{vmatrix}$$

with

$$F_i(z_i) = z_i \frac{\partial f_i}{\partial z_i}(z_i)/f_i(z_i), \quad i = 1, 2.$$

D_i stands for the differential operator with respect to z_i. Equivalently,

$$(3.7) \quad \tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}) = \left[\left(1 + \mu_1 \frac{\alpha_2 + \beta_2 k_2}{\lambda_2 + \mu_2 k_2} F_1(z_1)\right)\left(1 + \mu_2 \frac{\alpha_1 + \beta_1 k_1}{\lambda_1 + \mu_1 k_1} F_2(z_2)\right) - \frac{(\alpha_1\mu_1 - \beta_1\lambda_1)(\alpha_2\mu_2 - \beta_2\lambda_2)}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)} F_1(z_1)F_2(z_2)\right] f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}.$$

Proof: The proof is based on the method for treating Lagrange inversion problems introduced by the author [11]. For $i = 1, 2$, apply $(z_i D_i + (\lambda_i/\mu_i))$ on both sides of (3.4) to get

$$(3.8) \quad \left(z_i D_i + \frac{\lambda_i}{\mu_i}\right) f_{\underset{\sim}{k}}(\underset{\sim}{z}) = \left[\frac{\lambda_i + \mu_i k_i}{\mu_i} + (\lambda_i + \mu_i k_i)\left(\frac{\beta_{3-i}}{\mu_{3-i}} + \frac{\alpha_{3-i} - \frac{\beta_{3-i}}{\mu_{3-i}}\lambda_{3-i}}{\lambda_{3-i} + \mu_{3-i}k_{3-i}}\right) F_i(z_i)\right] f_{\underset{\sim}{k}}(\underset{\sim}{z}), \quad i = 1, 2.$$

Writing $c_i(\underset{\sim}{k}) = (\lambda_i + \mu_i k_i)^{-1}$, $i = 1, 2$, simple manipulations show that the system (3.8) of two equations is equivalent to the system

$$(3.9) \quad U_i f_{\underset{\sim}{k}}(\underset{\sim}{z}) = c_i(\underset{\sim}{k}) V f_{\underset{\sim}{k}}(\underset{\sim}{z}), \quad i = 1, 2,$$

where

$$U_i = \left(z_{3-i}D_{3-i} + \frac{\lambda_{3-i}}{\mu_{3-i}}\right)\left(\frac{1}{\mu_i} + \frac{\beta_{3-i}}{\mu_{3-i}} F_i(z_i)\right) + \left(\alpha_{3-i} - \frac{\beta_{3-i}}{\mu_{3-i}}\lambda_{3-i}\right) F_i(z_i)\left(\frac{1}{\mu_{3-i}} + \frac{\beta_i}{\mu_i} F_{3-i}(z_{3-i})\right)$$

and

$$(3.10) \quad V = \det \begin{vmatrix} z_1 D_1 + \frac{\lambda_1}{\mu_1} & \left(\alpha_2 - \frac{\beta_2}{\mu_2}\lambda_2\right) F_1(z_1) \\ \left(\alpha_1 - \frac{\beta_1}{\mu_1}\lambda_1\right) F_2(z_2) & z_2 D_2 + \frac{\lambda_2}{\mu_2} \end{vmatrix}.$$

Now Theorem 1 of [11] with $A = \mathbb{C}$, $M_1 = M_2 =$ set of fLs, U_i, V, $c_i(\underset{\sim}{k})$ as above, may be applied. The bilinear form we need is defined by

$$(3.11) \quad (a(\underset{\sim}{z}), b(\underset{\sim}{z})) = \langle \underset{\sim}{z}^0 \rangle a(\underset{\sim}{z}) b(\underset{\sim}{z}),$$

for fLs $a(\underset{\sim}{z})$ and $b(\underset{\sim}{z})$. Thus, by (4.4) of [11], the dual system

$$(3.12) \quad U_i^* h_{\underset{\sim}{k}}(\underset{\sim}{z}) = c_i(\underset{\sim}{k}) W h_{\underset{\sim}{k}}(\underset{\sim}{z}), \quad i = 1, 2,$$

[note that $V^* = W$ as defined in (3.6), since $(z_i D_i)^* = -z_i D_i$] has to be solved first. It is a simple matter of fact that (3.12), the dual of (3.9), is equivalent to the dual of (3.8), which reads

$$\left(-z_i D_i + \frac{\lambda_i}{\mu_i}\right) h_{\underset{\sim}{k}}(\underset{\sim}{z}) = \left[\frac{\lambda_i + \mu_i k_i}{\mu_i} + (\lambda_i + \mu_i k_i)\left(\frac{\beta_{3-i}}{\mu_{3-i}} + \frac{\alpha_{3-i} - \frac{\beta_{3-i}}{\mu_{3-i}}\lambda_{3-i}}{\lambda_{3-i} + \mu_{3-i}k_{3-i}}\right) F_i(z_i)\right] h_{\underset{\sim}{k}}(\underset{\sim}{z}), \quad i = 1, 2.$$

A solution of this system of equations is seen to be $h_{\underset{\sim}{k}}(\underset{\sim}{z}) = f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$, hence, by (4.6) of [11], respecting $V^* = W$,

$$\tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}) = \frac{\mu_1\mu_2}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)} Wh_{\underset{\sim}{k}}(\underset{\sim}{z}),$$

which establishes (3.5). A little bit of calculation from (3.5) leads to (3.7). □

Corollary 2 (Lagrange formula): Let $F = (f_{\underset{\sim}{k}}(\underset{\sim}{z}))_{\underset{\sim}{k} \in \mathbf{Z}^2}$ be as defined in (3.4). The coefficients in the expansion

$$(3.13) \quad g(\underset{\sim}{z}) = \sum_{\underset{\sim}{k} \in \mathbf{Z}^2} c_{\underset{\sim}{k}} f_{\underset{\sim}{k}}(\underset{\sim}{z})$$

are given by

$$(3.14) \quad c_{\underset{\sim}{k}} = \langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle g(\underset{\sim}{z}) \tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}),$$

with $\tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z})$ of (3.7), or

$$(3.15) \quad c_{\underset{\sim}{k}} = \frac{\mu_1\mu_2}{(\lambda_1 + \mu_1 k_1)(\lambda_2 + \mu_2 k_2)} \langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1} Vg(\underset{\sim}{z}).$$

Proof: Equation (3.14) is merely (3.2) for $f_{\underset{\sim}{k}}(\underset{\sim}{z})$ of (3.4), (3.15) is based on (3.14), (3.5), and $W^* = V$. □

As a first application, we shall prove (1.7) of Cohen & Hudson [5]. Take $f_1(t) = f_2(t) = \exp t$, which implies $F_1(t) = F_2(t) = t$. Let

$$g(\underset{\sim}{z}) = \frac{1}{\lambda_1\lambda_2} \sum_{j=0}^{\infty} \frac{\left[\left(\alpha_1 - \frac{\beta_1}{\mu_1}\lambda_1\right)\left(\alpha_2 - \frac{\beta_2}{\mu_2}\lambda_2\right) z_1 z_2\right]^j}{(\lambda_1/\mu_1 + 1)_j (\lambda_2/\mu_2 + 1)_j},$$

where $(a)_j = a(a + 1) \ldots (a + j - 1)$. For this choice of $f_i(t)$ [V depends on $F_i(t)$!], $g(\underset{\sim}{z})$ satisfies $Vg(\underset{\sim}{z}) = 1/\mu_1\mu_2$. Utilizing the Lagrange formula (3.15), from this fact we obtain

$$(3.16) \quad \frac{1}{\lambda_1\lambda_2} \sum_{j=0}^{\infty} \frac{\left[\left(\alpha_1 - \frac{\beta_1}{\mu_1}\lambda_1\right)\left(\alpha_2 - \frac{\beta_2}{\mu_2}\lambda_2\right) z_1 z_2\right]^j}{(\lambda_1/\mu_1 + 1)_j (\lambda_2/\mu_2 + 1)_j}$$

$$= \sum_{\underset{\sim}{k} \geq \underset{\sim}{0}} (\lambda_1 + \mu_1 k_1)^{-1} (\lambda_2 + \mu_2 k_2)^{-1} \frac{R_1(\underset{\sim}{k})^{k_1} R_2(\underset{\sim}{k})^{k_2}}{\underset{\sim}{k}!} (-\underset{\sim}{z})^{\underset{\sim}{k}} \exp(R_1(\underset{\sim}{k}) z_1 + R_2(\underset{\sim}{k}) z_2).$$

Equation (3.16) is another two-variable extension of (1.1) (set $z_2 = 0$).

4. Coefficient Formulas for Some Special Expansions

The following technical lemma turns out to be useful for further computations.

Lemma 3: Let $h(t)$ be an fLs in t. With the assumptions of Theorem 1,

$$(4.1) \quad \langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle h(z_{3-i}) F_i(z_i) f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$$

$$= -\frac{k_i(\lambda_{3-i} + \mu_{3-i} k_{3-i})}{(\lambda_i + \mu_i k_i)(\alpha_{3-i} + \beta_{3-i} k_{3-i})} \langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle h(z_{3-i}) f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}, \text{ for } i = 1, 2.$$

Proof: Without loss of generality we prove (4.1) for $i = 1$. We start with the identity

$$\langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle h(z_2) \left(1 + \mu_1 \frac{\alpha_2 + \beta_2 k_2}{\lambda_2 + \mu_2 k_2} F_1(z_1)\right) f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$$

$$= -\langle \underset{\sim}{z}^{\underset{\sim}{0}} \rangle h(z_2) z_2^{-k_2} f_2(z_2)^{-R_2(\underset{\sim}{k})} f_1(z_1)^{-\lambda_1(\alpha_2 + \beta_2 k_2)/(\lambda_2 + \mu_2 k_2)} \cdot$$

$$\cdot \frac{1}{k_1} z_1 D_1 \left[z_1^{-k_1} f_1(z_1)^{-\mu_1 k_1 (\alpha_2 + \beta_2 k_2)/(\lambda_2 + \mu_2 k_2)}\right].$$

Because of $(z_1D_1)^* = -z_1D_1$ [with respect to the bilinear form of (3.11)] the right-hand side of this equation is equal to

$$\frac{1}{k_1}\langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle z_1D_1\Big[h(z_2)z_2^{-k_2}f_2(z_2)^{-R_2(\underset{\sim}{k})}f_1(z_1)^{-\lambda_1(\alpha_2+\beta_2k_2)/(\lambda_2+\mu_2k_2)}\Big]\cdot$$
$$\cdot\ z_1^{-k_1}f_1(z_1)^{-\mu_1k_1(\alpha_2+\beta_2k_2)/(\lambda_2+\mu_2k_2)}.$$

Together with a bit of manipulation, we finally arrive at (4.1). □

Corollary 4: Let $f_{\underset{\sim}{k}}(\underset{\sim}{z})$ be given by (3.4). Then

$$(4.2)\qquad 1 = \sum_{\underset{\sim}{k}\geq\underset{\sim}{0}}\frac{\lambda_1\lambda_2 - k_1k_2\dfrac{(\alpha_1\mu_1-\beta_1\lambda_1)(\alpha_2\mu_2-\beta_2\lambda_2)}{(\alpha_1+\beta_1k_1)(\alpha_2+\beta_2k_2)}}{(\lambda_1+\mu_1k_1)(\lambda_2+\mu_2k_2)}d_{\underset{\sim}{k}}f_{\underset{\sim}{k}}(\underset{\sim}{z}),$$

where $d_{\underset{\sim}{k}} = \langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$.

Proof: By (3.14) we have to compute

$$\langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle 1\cdot\tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}) = \langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle\tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}).$$

Using the form (3.7) of $\tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z})$, repeated application of (4.1) gives

$$\langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle\tilde{f}_{\underset{\sim}{k}}(\underset{\sim}{z}) = \langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle\Big[\Big(1-\mu_1\frac{k_1}{\lambda_1+\mu_1k_1}\Big)\Big(1-\mu_2\frac{k_2}{\lambda_2+\mu_2k_2}\Big)$$
$$-\frac{(\alpha_1\mu_1-\beta_1\lambda_1)(\alpha_2\mu_2-\beta_2\lambda_2)}{(\alpha_1+\beta_1k_1)(\alpha_2+\beta_2k_2)}\cdot\frac{k_1k_2}{(\lambda_1+\mu_1k_1)(\lambda_2+\mu_2k_2)}\Big]f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$$
$$=\frac{\lambda_1\lambda_2 - k_1k_2\dfrac{(\alpha_1\mu_1-\beta_1\lambda_1)(\alpha_2\mu_2-\beta_2\lambda_2)}{(\alpha_1+\beta_1k_1)(\alpha_2+\beta_2k_2)}}{(\lambda_1+\mu_1k_1)(\lambda_2+\mu_2k_2)}\langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1},$$

which furnishes (4.2). □

For $f_i(z_i) = \exp(-z_i)$ and $f_i(z_i) = (1+z_i)^{-1}$, respectively, the expansions (2.8) and (2.9) are obtained as special cases of (4.2). The mixed analogue (2.10) is (4.2) with $f_1(z_1) = \exp(-z_1)$ and $f_2(z_2) = (1+z_2)^{-1}$.

Quite analogously, we prove

Corollary 5: If $f_{\underset{\sim}{k}}(\underset{\sim}{z}) = \underset{\sim}{z}^{\underset{\sim}{k}}f_1(z_1)^{r_1(\underset{\sim}{k})}f_2(z_2)^{R_2(\underset{\sim}{k})}$, then

$$(4.3)\qquad \frac{1}{\lambda_1+\mu_2\alpha_1F_2(z_2)} = \sum_{\underset{\sim}{k}\geq\underset{\sim}{0}}(\lambda_1+\mu_1k_1)^{-1}d_{\underset{\sim}{k}}f_{\underset{\sim}{k}}(\underset{\sim}{z}),$$

where $d_{\underset{\sim}{k}} = \langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$.

Sketch of Proof: Again using (3.14), (3.7), and (4.1) we proceed along the lines of the proof of the preceding corollary. □

The expansions (2.4)-(2.7) are special cases of (4.3). Finally, we have

Corollary 6: If $f_{\underset{\sim}{k}}(\underset{\sim}{z}) = \underset{\sim}{z}^{\underset{\sim}{k}}f_1(z_1)^{r_1(\underset{\sim}{k})}f_2(z_2)^{r_2(\underset{\sim}{k})}$, then

$$(4.4)\qquad \frac{1-\dfrac{\mu_1\mu_2}{\lambda_1\lambda_2}F_1(z_1)F_2(z_2)}{\Big(1+\dfrac{\mu_1}{\lambda_2}F_1(z_1)\Big)\Big(1+\dfrac{\mu_2}{\lambda_1}F_2(z_2)\Big)} = \sum_{\underset{\sim}{k}\geq\underset{\sim}{0}}d_{\underset{\sim}{k}}f_{\underset{\sim}{k}}(\underset{\sim}{z}),$$

where $d_{\underset{\sim}{k}} = \langle \underset{\sim}{z}^{\underset{\sim}{0}}\rangle f_{\underset{\sim}{k}}(\underset{\sim}{z})^{-1}$.

Proof: Observe that the left-hand side of (4.4) is equal to

$$1/\left(1 + \frac{\mu_1}{\lambda_2} F_1(z_1)\right) + 1/\left(1 + \frac{\mu_2}{\lambda_1} F_2(z_2)\right) - 1.$$

This in hand, the method used to prove Corollary 4 can be used again to settle (4.4). □

Equations (2.1)-(2.3) are special cases of (4.4).

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