LENGTH OF THE n-NUMBER GAME

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The n-number game is defined as follows. Let $S = (s_1, s_2, \ldots, s_n)$ be an n-tuple of nonnegative integers. A new n-tuple $D(S) = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n)$ is obtained by taking numerical differences; that is, $\hat{s}_i = |s_i - s_{i+1}|$. Subscripts are reduced modulo n so that $\hat{s}_n = |s_n - s_1|$. The sequence S, $D(S)$, $D^2(S)$, ... is called *the n-number game generated by S*. To see that a game contains only a finite number of distinct tuples let $|S| = \max\{s_i\}$ and observe that $|S| \geq |D(S)|$. Since there are only a finite number of n-tuples with entries less than or equal to $|S|$, eventually repetition must take place. When $n = 2^w$, it is well known that every game terminates with $(0, 0, \ldots, 0)$. That this is not the case for other values of n is easily seen by considering the following 3-tuple:

$$\begin{aligned} R &= (1, 0, 0) \\ D(R) &= (1, 0, 1) \\ D^2(R) &= (1, 1, 0) \\ D^3(R) &= (0, 1, 1) \\ D^4(R) &= (1, 0, 1) = D(R) \end{aligned}$$

The tuples $D(R)$, $D^2(R)$, and $D^3(R)$ form what is called a *cycle*.

For any n-tuple S, we say the game generated by S has *length* λ, denoted by $L(S)$, if $D^\lambda(S)$ is in a cycle, but $D^{\lambda-1}(S)$ is not. Thus, in the example above, $L(R) = 1$, while $L(D(R)) = 0$. For each n, the length of games is unbounded. That is, for any λ, there exists an n-tuple S such that $L(S) > \lambda$. On the other hand, for tuples S with $|S| \leq m$, there is a game of maximum length, since there are only a finite number of such tuples. We introduce the following notation:

$$\begin{aligned} \mathscr{S}_n(m) &= \{S \mid S \text{ is an } n\text{-tuple with } |S| = m\}, \\ \mathscr{L}_n(m) &= \max\{L(S) \mid S \in \mathscr{S}_n(m)\}. \end{aligned}$$

On occasion, when the context is clear, we will drop the subscript. The values of $\mathscr{L}_4(m)$ and $\mathscr{L}_7(m)$, along with tuples giving games of maximum lengths, have been determined in [10] and [6]. We consider this question when n is not a power of 2. We first find an upper bound on $\mathscr{L}_n(m)$. Then we show that this bound is actually realized when $n = 2^w + 1$.

Before proceeding, a few additional comments are in order. Observe that, for any tuple S, if we multiply all the entries by a constant c and denote the resulting tuple by cS, then

$$(1) \qquad D(cS) = cD(S).$$

Additionally, if all the nonzero entries of S are equal with $S \in \mathscr{S}(m)$, then $S = mE$ for some $E \in \mathscr{S}(1)$. In particular, an entry e_i in E equals 1 if and only if the corresponding entry in S, s_i, equals m.

Since a game concludes when a cycle is reached, it is important to be able to identify those tuples which occur in a cycle. This author did that in [5]. The following theorem gives the salient facts from that work. We say that an n-tuple S has *a predecessor* if $S = D(R)$ for some n-tuple R.

Theorem 1: Let $n = kr$ where $k = 2^k$ and r is odd with $r > 1$. Suppose S is an n-tuple. Then

(i) S has a predecessor if and only if there exist values $\varepsilon_\ell \in \{-1, 1\}$, $\ell = 1, 2, n$, such that

$$\sum_{\ell=1}^{n} \varepsilon_\ell s_\ell = 0.$$

(ii) S is in a cycle if and only if all its entries are 0 or $|S|$ and

$$\sum_{j=0}^{r-1} e_{i+jk} \equiv 0 \pmod 2), \text{ for } i = 1, \ldots, k,$$

where $S = |S| \cdot E$ with $E = (e_1, e_2, \ldots, e_n) \in \mathscr{S}(1)$.

Part (ii) guarantees that when n is not a power of 2, there are nontrivial cycles; indeed, for n odd, the tuple $\mathbb{E} = (0, \ldots, 0, 1, 1)$ is in a cycle. Moreover, (ii) along with (1) gives

(2) $\quad L(cS) = L(S)$.

2. A Bound on $\mathscr{L}_n(m)$

For $S \in \mathscr{S}_n(m)$, we say that S *has* μ *0's and m's in a row*, denoted by $\mu(S)$, if the following conditions are met: there exists an integer η such that $s_i \in \{0, m\}$ for $i = \eta, \eta + 1, \ldots, \eta + \mu - 1$, at least one of these s_i equals m, and μ is as large as possible. As usual, we reduce subscripts modulo n. Thus, for example,

$$\mu(S) = 6 \text{ when } S = (3, 2, 3, 0, 1, 3, 0, 3, 0, 0).$$

Loosely speaking, a tuple S will produce a long game if, at each step, $\mu(D^k(S))$ is as large as possible. In determining an upper bound on $\mathscr{L}_n(m)$, the following lemmas will be useful.

Lemma 1: Let $S \in \mathscr{S}_n(m)$, $\mu(S) = t$, and $t < n$. If $D(S) \in \mathscr{S}_n(m)$, then $\mu(D(S)) \le t - 1$.

Proof: By hypothesis, for some η, we have

$$\begin{aligned} &s_i \in \{0, m\} && \text{for } i = \eta, \eta + 1, \ldots, \eta + t - 1,\\ &s_i = m && \text{for some } i,\ \eta \le i \le \eta + t - 1,\\ &1 \le s_{\eta-1},\ s_{\eta+t} \le m - 1. \end{aligned}$$

As before, let $D(S) = (\hat{s}_1, \ldots, \hat{s}_n)$. Then

$$\begin{aligned} &\hat{s}_i \in \{0, m\} && \text{for } i = \eta, \eta + 1, \ldots, \eta + t - 2,\\ &1 \le \hat{s}_{\eta-1},\ \hat{s}_{\eta+t-1} \le m - 1. \end{aligned}$$

Hence, if $|D(S)| = m$, then $\mu(D(S)) \le t - 1$. □

At first glance, it might seem in Lemma 1 that, if $|D(S)| = m$, then $\mu(D(S))$ must equal $t - 1$. It is possible, however, to have strict inequality. This would occur if $\hat{s}_i = 0$ for $\eta \le i \le \eta + t - 2$, while $\hat{s}_j = m$ for some other j.

Lemma 2: Suppose that $S \in \mathscr{S}_n(m)$ and not all the nonzero entries equal m. Then $|D^{n-1}(S)| \le m - 1$. Further, if S has a predecessor, then $|D^{n-2}(S)| \le m - 1$.

Proof: Let $\mu(S) = t$. By hypothesis, $t \le n - 1$, and if S has a predecessor, then by Theorem 1(i), $t \le n - 2$. In either case, Lemma 1 applies. So, if $|D^i(S)| = m$, for $i = 1, \ldots, t - 1$, then $\mu(D^i(S)) \le t - i$. Of course, if $\mu(D^j(S)) = 1$, then $|D^{j+1}(S)| \le m - 1$. Thus, $|D^t(S)| \le m - 1$. □

In a moment we will consider those tuples in which all nonzero entries equal m. In that case, $S = mE$ for some $E \in \mathscr{S}(1)$. For tuples in $\mathscr{S}_n(1)$, the following is useful. Let $\mathbb{A} = \mathbb{Z}_2[t]/\mathscr{I}$ where $\mathbb{Z}_2[t]$ is the polynomial ring over

$\mathbb{Z}_2$ and $\mathscr{I}$ is the principal ideal generated by $t^n + 1$. We associate with $E = (e_1, \ldots, e_n) \in \mathscr{S}_n(1)$, the polynomial

$$\mathscr{P}_E(t) = e_n + e_{n-1}t + \cdots + e_2 t^{n-2} + e_1 t^{n-1} \text{ in } \mathbb{A}.$$

Since $\hat{e}_i = |e_i - e_{i+1}| = e_i + e_{i+1}$ in $\mathbb{Z}_2$ and $t^n = 1$ in $\mathbb{A}$,

$$\begin{aligned}(3) \qquad \mathscr{P}_{D(E)}(t) &= (e_n + e_1) + (e_{n-1} + e_n)t + \cdots + (e_2 + e_3)t^{n-2} + (e_1 + e_2)t^{n-1} \\ &= (1 + t)\mathscr{P}_E(t).\end{aligned}$$

Lemma 3: Let $n = kr$, where $k = 2^k$ and r is odd with $r > 1$. Suppose $S \in \mathscr{S}_n(m)$ and all the nonzero entries equal m. Then $L(S) \le k$. Further, if S has a predecessor, then $L(S) \le k - 1$.

Proof: As usual, we let $S = mE$, where $E = (e_1, \ldots, e_n) \in \mathscr{S}(1)$. For the first part, by (2), we need only show that $D^k(E)$ is in a cycle. Using (3), we find

$$\begin{aligned}\mathscr{P}_{D^k(E)}(t) &= (1 + t)^k \mathscr{P}_E(t) \\ &= (1 + t^k)\mathscr{P}_E(t) \\ &= (1 + t^k)(e_n + e_{n-1}t + \cdots + e_2 t^{n-2} + e_1 t^{n-1}) \\ &= \sum_{\ell=0}^{k-1}(e_{n-\ell} + e_{k-\ell})t^\ell + \sum_{\ell=k}^{n-1}(e_{n-\ell} + e_{n+k-\ell})t^\ell \text{ in } \mathbb{A}.\end{aligned}$$

The second equality holds since k is a power of 2 and so all the binomial coefficients in $(1 + t)^k$ except for the first and last are even. From the above, we see that

$$D^k(E) = (e_1 + e_{k+1},\ e_2 + e_{k+2},\ \ldots,\ e_{n-k} + e_n,\ e_{n-k+1} + e_1,\ \ldots,\ e_n + e_k).$$

We now check condition (ii) of Theorem 1. In doing so, we use the fact that $n - k = (r - 1)k$. For $i = 1$, we have

$$(e_1 + e_{k+1}) + (e_{k+1} + e_{2k+1}) + \cdots + (e_{n-k+1} + e_1) \equiv 0 \pmod 2.$$

Similarly, (ii) holds for all other values of i. Thus, $D^k(E)$ is in a cycle and $L(E) \le k$.

For the second part, it is also sufficient to show that $L(E) \le k - 1$. Consider the tuple $F = (f_1, f_2, \ldots, f_n) \in \mathscr{S}(1)$ defined by

$$f_1 = 0,\ f_i = e_1 + e_2 + \cdots + e_{i-1} \pmod 2,\ i = 2, \ldots, n.$$

Since S has a predecessor, E does as well; because the entries of E are either 0 or 1, Theorem 1(i) gives

$$e_1 + e_2 + \cdots + e_n \equiv 0 \pmod 2.$$

This means that $f_n = e_n$ and so $D(F) = E$. Thus, $L(E) = L(D(F)) \le k - 1$. □

Theorem 2: Let $n = kr$, where $k = 2^k$ and r is odd with $r > 1$. Then $\mathscr{L}_n(m) \le (m - 1)(n - 2) + k$.

Proof: Let $S \in \mathscr{S}_n(m)$. If all the nonzero entries of S are equal, then by Lemma 3, $L(S) \le k$ and so the theorem holds. Otherwise, by Lemma 2, $|D^{n-1}(S)| \le m - 1$. Continuing, suppose that, for some $\ell = 1, \ldots, m - 2$, all the nonzero entries of $D^{\ell(n-2)+1}(S)$ are equal. Then, again by Lemma 3, $L(D^{\ell(n-2)+1}(S)) \le k - 1$, which means $L(S) \le \ell(n - 2) + k$. On the other hand, if the latter condition does not hold, then, by Lemma 2, $|D^{(m-1)(n-2)+1}(S)| \le 1$. Another application of Lemma 3 gives the desires result. □

If there is a tuple $S \in \mathscr{S}_n(m)$ with $L(S) = (m - 1)(n - 2) + k$, then the proof of Theorem 2 tells us what the tuples in the game must look like.

Corollary 1: Let $n = kr$, where $k = 2^\kappa$ and r is odd with $r > 1$. If

$$\mathscr{L}_n(m) = (m-1)(n-2) + k,$$

then there exists $S \in \mathscr{S}_n(m)$ such that

(i) $|D^{\ell(n-2)+1}(S)| = m - \ell$ and $\mu(D^{\ell(n-2)+1}(S)) = n - 2$ for $\ell = 0, \ldots, m-1$,

(ii) $L(D^{(m-1)(n-2)+1}(S)) = k - 1$.

Proof: This follows immediately from the proof of Theorem 2. □

In a moment we will state a condition for the existence of a game of maximum length in terms of the n-tuple $(0, \ldots, 0, 1, 1)$. Before proceeding, two comments are in order. First, if the entries of an n-tuple are rearranged so that adjacent elements remain adjacent, then similar games result. Or, more precisely, if $S = (s_1, s_2, \ldots, s_n)$ and σ_1 is a permutation contained in the dihedral group $\mathscr{D}_n$, then

(4) $\quad D(\sigma_1(S)) = \sigma_2(D(S))$ for some $\sigma_2 \in \mathscr{D}_n$.

Second, it is convenient to associate with $S = (s_1, s_2, \ldots, s_n)$ an n-tuple $\mathscr{M}(S) \in \mathscr{S}(1)$ which is related to the parity of the entries of S. We define $\mathscr{M}(S) = (m_1, m_2, \ldots, m_n)$ in the obvious way with $m_i \equiv s_i \pmod 2$. Observe that

(5) $\quad \mathscr{M}(D(S)) = D(\mathscr{M}(S))$.

Theorem 3: Let $n = kr$, where $k = 2^\kappa$ and r is odd with $r > 1$. Suppose for $m \geq 4$, $\mathscr{L}_n(m) = (m-1)(n-2) + k$. Then, for some $\sigma \in \mathscr{D}_n$,

$$D^{2(n-2)}(\mathbb{E}) = \sigma(\mathbb{E}), \text{ where } \mathbb{E} = (0, \ldots, 0, 1, 1).$$

Proof: By hypothesis, there exists an n-tuple S with $|S| = m$ and

$$L(S) = (m-1)(n-2) + k.$$

Let $T = D^{(m-4)(n-2)+1}(S)$. Corollary 1 implies that

$$|T| = |D^{(m-4)(n-2)+1}(S)| = 4,\ \mu(T) = n - 2, \text{ and } |D^{(n-2)}(T)| = 3.$$

Since $\mu(T) = n - 2$, T has exactly two adjacent entries with values in $\{1,2,3\}$. One of these must equal either 1 or 3; for, if not, then $|D^{(n-2)}(T)| \leq 2$. Moreover, since T has a predecessor, Theorem 1(i) guarantees that both are in $\{1,3\}$. This shows that

$$\mathscr{M}(T) = \sigma_1(\mathbb{E}) \text{ for some } \sigma_1 \in \mathscr{D}_n.$$

Similarly,

$$\mathscr{M}(D^{2(n-2)}(T)) = \sigma_2(\mathbb{E}) \text{ for } \sigma_2 \in \mathscr{D}_n.$$

Hence,

$$\begin{aligned}\sigma_2(\mathbb{E}) &= \mathscr{M}(D^{2(n-2)}(T)) \\ &= D^{2(n-2)}(\mathscr{M}(T)) \\ &= D^{2(n-2)}(\sigma_1(\mathbb{E})) \\ &= \sigma_3(D^{2(n-2)}(\mathbb{E}))\end{aligned}$$

The second equality follows from (5); the last, from (4). Thus, for $\sigma = \sigma_3^{-1}\sigma_2 \in \mathscr{D}_n$, $D^{2(n-2)}(\mathbb{E}) = \sigma(\mathbb{E})$. □

Theorem 3 is the heart of the matter. Whether or not there exists an n-tuple which has the maximum possible length depends in large part on $\mathbb{E}$. Since $\mathbb{E} \in \mathscr{S}(1)$, Theorem 3 can be recast in terms of polynomials in $\mathbb{A}$. Using (3), we see that, in order to have an n-tuple of maximum length,

$$(1+t)^{2(n-2)}\mathscr{P}_{\mathbb{E}}(t) = \mathscr{P}_{\sigma(\mathbb{E})}(t).$$

Since $\mathscr{P}_{\mathbb{E}}(t) = 1 + t$, $\mathscr{P}_{\sigma(\mathbb{E})}(t) = t^j + t^{j+1}$ for some j, where, if necessary, the exponent $j + 1$ is reduced modulo n. Thus, we have

Corollary 2: Let $n = kr$, where $k = 2^\kappa$ and r is odd with $r > 1$. Suppose that, for $m \geq 4$, $\mathscr{L}_n(m) = (m-1)(n-2) + k$. Then, for some j,

$$(1+t)^{2n-3} = t^j(1+t) \tag{6}$$

in $\mathbb{A}$. □

Theorem 4: Let n be an integer such that $n \neq 2^w$ and $n \neq 2^w + 1$ for any w. Then, for $m \geq 4$, $\mathscr{L}_n(m) < (m-1)(n-2) + k$.

Proof: First, suppose n is even. By Theorem 1(ii), $\mathbb{E} = (0, \ldots, 0, 1, 1)$ is not in a cycle. Thus, $D^i(\mathbb{E}) \neq \mathbb{E}$ for any i. Now, if $D^{2(n-2)}(\mathbb{E}) = \sigma(\mathbb{E})$ for some $\sigma \in \mathscr{D}_n$, then $D^{2(n-2)p}(\mathbb{E}) = \mathbb{E}$ where p is the order of σ in $\mathscr{D}_n$. Consequently, the conclusion of Theorem 3 cannot hold.

For n odd, we will expand $(1+t)^{2n-3}$ denoting the ℓ^{th} binomial coefficient by c_ℓ.

$$\begin{aligned}(1+t)^{2n-3} &= \sum_{\ell=0}^{2n-3} c_\ell t^\ell = \sum_{\ell=0}^{n-3}(c_\ell + c_{\ell+n})t^\ell + (c_{n-2}t^{n-2} + c_{n-1}t^{n-1}) \\ &= \sum_{\ell=0}^{n-3}(c_\ell + c_{n-3-\ell})t^\ell + (c_{n-2}t^{n-2} + c_{n-2}t^{n-1}) \\ &= \sum_{\ell=0}^{\frac{n-5}{2}}(c_\ell + c_{n-3-\ell})(t^\ell + t^{n-3-\ell}) \\ &\qquad + 2c_{\frac{n-3}{2}}t^{\frac{n-3}{2}} + c_{n-2}(t^{n-2} + t^{n-1}).\end{aligned}$$

The second equality follows by using $t^n = 1$; the third, from $c_\ell = c_{2n-3-\ell}$. Now when $2n - 3 = 2^v - 1$ for some v, all the binomial coefficients are odd, so that we have

$$(1+t)^{2n-3} = t^{n-2}(1+t).$$

Thus, (6) holds for $n = 2^w + 1$, where $w = v - 1$. On the other hand, when $2n - 3 \neq 2^v - 1$ for any v, then c_{n-2} is even. So, if t^ℓ is present in the expansion of $(1+t)^{2n-3}$, then so also is $t^{n-3-\ell}$. Hence, (6) cannot hold. □

3. The Case $n = 2^w + 1$

We now consider the case in which $n = 2^w + 1$. Corollary 2 and Theorem 4 imply that a game of maximum length is possible. We show that this actually occurs. Before examining the general case, we consider the special case $n = 3$.

Lemma 4: Let $n = 3$ and define $T_m = (m-1, 1, m)$. Then, for $m \geq 2$, $D(T_m) = \sigma(T_{m-1})$ for some $\sigma \in \mathscr{D}_3$.

Proof: The result is immediate since $D(T_m) = (m-2, m-1, 1)$. □

Lemma 5: Suppose $n = 2^w + 1$, $w \geq 2$. Let $T_m = (0, 0, \ldots, 0, m-1, 1, m)$. Then, for $m \geq 2$,

$$\begin{aligned}D^{n-4}(T_m) &= (0, m-1, t_3, t_4, \ldots, t_{n-1}, m) \\ D^{n-3}(T_m) &= (m-1, 1, \ldots, 1, m) \\ D^{n-2}(T_m) &= (m-2, 0, \ldots, 0, m-1, 1)\end{aligned}$$

where the entries in $D^{n-4}(T_m)$ have the property that $|t_i - t_{i+1}| = 1$ for $i = 2, \ldots, n-1$.

Proof: The proof proceeds by induction on w. Suppose that $w = 2$ so that $n = 5$. Then $T_m = (0, 0, m - 1, 1, m)$ and it is easily seen that $D(T_m) = (0, m - 1, m - 2, m - 1, m)$.

Suppose that the Lemma 5 holds for $w - 1$; more specifically, suppose that

$$D^{\ell-4}(\overline{T}_m) = (0, m - 1, r_3, r_4, r_{\ell-1}, m), \text{ and}$$

$$D^{\ell-4}(\overline{T}_{m-1}) = (0, m - 2, s_3, s_4, \ldots, s_{\ell-1}, m - 1),$$

where $\ell = 2^{w-1} + 1$, $|r_i - r_{i+1}| = 1$, and $|s_i - s_{i+1}| = 1$ for $i = 2, \ldots, \ell - 1$. Consider the $(2^w + 1)$-tuple T_m. We can view T_m as a 2^{w-1} zero-tuple concatenated with a $2^{w-1} + 1$ "T_m-type-tuple." Thus, when we compute $D^k(T_m)$ for k less than $2^{w-1} - 2$, we have the same pattern we have for the $2^{w-1} + 1$ case. Thus, we have

k	$D\ (T\ )$
$2^{w-1} - 3$	$(0, 0, \ldots, 0, 0, 0, m - 1, r_3, r_4, \ldots, r_{\ell-1}, m)$
$2^{w-1} - 2$	$(0, 0, \ldots, 0, 0, m - 1, 1, 1, 1, \ldots, 1, m)$
$2^{w-1} - 1$	$(0, 0, \ldots, 0, m - 1, m - 2, 0, 0, \ldots, 0, m - 1, m)$
2^{w-1}	$(0, 0, \ldots, m - 1, 1, m - 2, 0, \ldots, 0, m - 1, 1, m)$
$\vdots$	$\vdots$
$2^w - 3$	$(0, m - 1, s_{\ell-1}, \ldots, s_3, m - 2, m - 1, r_3, r_4, \ldots, r_{\ell-1}, m)$.

Note that for $k = 2^{w-1}$, $D^k(T_m)$ may be viewed as the $2^{w-1} + 1$ "T_{m-1}-type" tuple, $(0, \ldots, 0, m - 1, 1, m - 2)$, concatenated with the 2^{w-1} tuple, $(0, \ldots, 0, m - 1, 1, m)$. The latter is like the $2^{w-1} + 1$ "T_m-type" tuple, except that it is missing the leading zero. By induction, the second through $(n - 1)^{\text{st}}$ entries in $D^k(T_m)$, $k = 2^w - 3 = n - 4$, differ from the next one by 1. Thus, $D^{n-4}(T_m)$ has the proper form. The conclusion for $D^{n-3}(T_m)$ and $D^{n-2}(T_m)$ follows immediately. $\square$

Theorem 5: Suppose $n = 2^w + 1$ for $w \geq 1$. Define $R_m = (0, 0, \ldots, 0, m - 1, m)$ for $m \geq 1$. Then $L(R_m) = (m - 1)(n - 2) + 1$.

Proof: Note that $D(R_m) = T_m = (0, \ldots, 0, m - 1, 1, m)$. Now, by Lemmas 4 and 5, $D^{n-2}(T_m) = \sigma(T_{m-1})$ for some $\sigma \in \mathscr{D}_n$ and $m \geq 2$. Further, T_1 is contained in a cycle, but no other T_m is. Thus, we have $L(R_m) = (m - 1)(n - 2) + 1$.

4. Remaining Questions

For n not a power of 2 and $n \neq 2^w + 1$, how large is $\mathscr{L}_n(m)$? What tuple produces the longest game? Only for $n = 7$ are the answers to these questions known [6].

Because Theorem 3 cannot hold for even n, it is tempting to try to prove a related version using $\mathbb{E} = (0, \ldots, 0, 1, 0, \ldots, 0, 1)$, where the 1's occur in the $(n - k)^{\text{th}}$ and n^{th} places. All efforts to date have failed. What relation, if any, does $\mathscr{L}_{2n}(m)$ have to $\mathscr{L}_n(m)$? The following is a limited answer to that question.

Theorem 6: $2\mathscr{L}_n(m) \leq \mathscr{L}_{2n}(m)$.

Proof: Let $S \in \mathscr{S}_n(m)$ with $L(S) = \mathscr{L}_n(m)$. Then the tuple $S \wedge 0$, where

$$S \wedge 0 = (0, s_1, 0, s_2, 0, s_3, \ldots, 0, s_n)$$

is in $\mathscr{S}_{2n}(m)$. By Theorem 1(ii), $D(S \wedge 0)$ is in a cycle if and only if S is. Further, $D^2(S \wedge 0) = D(S) \wedge 0$. Thus, $L(S \wedge 0) = 2L(S)$. $\square$

Unfortunately, from the few cases studied, it appears that the above inequality is a strict one.

The n-number game has been studied extensively; indeed, many key results keep reappearing in the literature and being reproved. An extensive bibliography appears in [7]. In the interest of completeness, additional references which either do not appear in that article or were published after 1982 are listed below.

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