ON A HOGGATT-BERGUM PAPER WITH TOTIENT FUNCTION APPROACH FOR DIVISIBILITY AND CONGRUENCE RELATIONS

Sahib Singh
Clarion University of Pennsylvania, Clarion, PA 16214
(Submitted August 1988)

During their discussion of divisibility and congruence relations of the Fibonacci and Lucas numbers, Hoggatt & Bergum found values of n satisfying the congruences $F_n \equiv 0 \pmod{n}$ or $L_n \equiv 0 \pmod{n}$. In this connection, Hoggatt & Bergum's research appears in Theorems 1, 3, 5, 6, and 7 of [4]. The present paper originated on the same lines in search of values of n that satisfy $\phi(F_n) \equiv 0 \pmod{n}$ or $\phi(L_n) \equiv 0 \pmod{n}$, where ϕ is the totient function. Before going into the analysis of the problem, we state some results that will be quoted frequently.

(a) (i) For $n > 2$, $\phi(n)$ is even; (ii) if $m|n$, then $\phi(m)|\phi(n)$ ([3], pp. 140-41).
(b) All odd prime divisors of L_{2n+1} are of the form $10m \pm 1$ ([4], p. 193).
(c) Every F_n with $n > 12$ and L_n with $n > 6$ has at least one primitive prime divisor ([6], p. 15).
(d) Let p be a primitive prime divisor of F_n $(n > 5)$; if $n \equiv 5 \pmod{10}$, then $p \equiv 1 \pmod{4n}$ ([6], p. 10).
(e) A primitive prime divisor p of L_{5n} with $n \geq 1$ satisfies $p \equiv 1 \pmod{10n}$ ([6], p. 11).
(f) Let n be odd and p an odd primitive prime divisor of F_n; if $p \equiv \pm 1 \pmod{10}$, then $p \equiv 1 \pmod{4n}$ ([1], p. 254).
(g) Let p be an odd primitive prime divisor of L_n; if $p \equiv \pm 1 \pmod{10}$, then $p \equiv 1 \pmod{2n}$ ([1], p. 255).

We begin our discussion by proving the following theorem.

Theorem 1: If n is an odd integer greater than 3, then
(i) $\phi(L_n) \equiv 0 \pmod{n}$;
(ii) $\phi(F_{2n}) \equiv 0 \pmod{2n}$.

Proof: Both results are true when $n = 5$. Thus, we choose $n \geq 7$.

(i) Based on (b) and (c), we have the existence of at least one primitive prime divisor p of L_n of the form $10m \pm 1$. Consequently, by (g):

(1) $\quad p \equiv 1 \pmod{2n}$.

Since $p|L_n$ and $\phi(p) \equiv 0 \pmod{2n}$ is true from (1), we have, using (a),

$$\phi(L_n) \equiv 0 \pmod{2n} \Rightarrow \phi(L_n) \equiv 0 \pmod{n}$$

(ii) Since $F_{2n} = F_n L_n$ and $\phi(L_n) \equiv 0 \pmod{2n}$, we have $2n|\phi(F_{2n})$.

Note: From the above, with odd $n > 3$, $\phi(L_n) \equiv 0 \pmod{2n}$ and $\phi(F_{2n}) \equiv 0 \pmod{4n}$ are both true. The second part follows from [5].

Corollary: If n is odd, $n > 3$, and $3|n$, then $4n|\phi(L_n)$.

Proof: By Lemma 1 of [4] (p. 193), $4|L_n$. From Theorem 1, $p|L_n$, where $p \equiv 1 \pmod{2n}$; consequently, by (a), $\phi(4p)|\phi(L_n)$. This proves our result.

In regard to Fibonacci numbers with even subscripts, we prove the following theorem.

Theorem 2: The congruence $\phi(F_{2N}) \equiv 0 \pmod{2N}$ is true for all positive integers N, except when $N = 1, 2, 3, 4, 8, 16$.

Proof: It is easy to verify that, for $N = 1, 2, 4, 8, 16$, the congruence $\phi(F_{2N}) \equiv N \pmod{2N}$ holds and, for $N = 3$, the result $\phi(F_6) \equiv 4 \pmod{6}$ is true. Excluding these values, we complete the proof by considering the following cases:

Case 1. If N is odd and greater than 3, the result follows from Theorem 1.

Case 2. For even values of N, we discuss the proof in two parts:

Part 1. Let $N = 2^{n-1}$, $n \geq 6$. For $n = 6$, the result

$$\phi(F_{64}) \equiv 0 \pmod{64}$$

is true (see [1], App. Table). For $n > 6$, we apply induction on n.

$\phi(F_{2^n}) \equiv 0 \pmod{2^n}$ is true by inductive hypothesis.

$\phi(L_{2^n}) \equiv 0 \pmod{2}$ is true by (a), and

$\phi(F_{2^{n+1}}) = \phi(F_{2^n})\phi(L_{2^n})$; therefore,

$\phi(F_{2^{n+1}}) \equiv 0 \pmod{2^{n+1}}$.

Part 2. Let $N = 2^{n-1}t$, where $n \geq 1$, t is odd, $t > 1$, and $2N \neq 6$. If $n = 1$, see Theorem 1. If $n > 1$, we use induction on n.

$F_{2^{n+1}\cdot t} = F_{2^n\cdot t}L_{2^n\cdot t}$; $L_{2^n} | L_{2^n\cdot t}$ and L_{2^n} & $F_{2^n\cdot t}$ are relatively prime; therefore,

$\phi(F_{2^n\cdot t}) \cdot \phi(L_{2^n})$ divides $\phi(F_{2^{n+1}\cdot t})$.

Repeating the argument of Part 1 above, we observe that

$\phi(F_{2^n\cdot t}) \equiv 0 \pmod{2^n \cdot t}$

is true by the inductive hypothesis,

$\phi(L_{2^n}) \equiv 0 \pmod{2}$

follows from (a). Hence, the proof is complete.

For examination of Lucas numbers with even subscripts, it is important to study the values of $\phi(L_{2^n})$. By verification, it follows that $\phi(L_{2^n}) \equiv 0 \pmod{2^n}$ is true when $n = 1, 5, 6, 7, 8$ and false for $n = 2, 3, 4$. It remains an open question whether $\phi(L_{2^n}) \equiv 0 \pmod{2^n}$ would be true for all $n \geq 5$ or for infinitely many n or only for a finite number n.

Since $2^n \equiv 0 \pmod{4}$ is true for $n \geq 5$, every odd prime divisor of L_{2^n} is one of the forms $40m + 1$, $40m + 7$, $40m + 9$, $40m + 23$ ([6], p. 11). In [7] it is proved that $L_{2^n} \equiv 3 \pmod{4}$ and, hence, contains an odd number of primes of the form $4m + 3$.

In view of this, we conclude that, if L_{2^n} is the product of an even number of primes, then it must contain at least one prime p of the type $40m + 1$ or $40m + 9$. If this prime p is primitive, then $p \equiv 1 \pmod{2^{n+1}}$ by (g). In this case, $2^n | \phi(p)$ and, consequently, $\phi(L_{2^n}) \equiv 0 \pmod{2^n}$ is true.

Based on this discussion, we are led to make the following conjecture.

Conjecture: There may exist infinitely many n such that $\phi(L_{2^n}) \equiv 0 \pmod{2^n}$ is true.

It is interesting to note that the following allied result holds.

Theorem 3: For all positive integers n, $\phi(L_{2^n} + 1) \equiv 0 \pmod{2^n}$ is true.

Proof: Using the Binet form, it is easy to see that

$$L_{2^{n+1}} + 1 = (L_{2^n} + 1)(L_{2^n} - 1).$$

Since $(L_{2^n} - 1)$ is always even, it follows by induction that $L_{2^n} + 1 \equiv 0 \pmod{2^{n+1}}$ is always true. Hence, $2^n = \phi(2^{n+1})$ divides $\phi(L_{2^n} + 1)$.

Some special cases are discussed in the following theorems.

Theorem 4: If n is a positive integer, the following congruences hold:

(i) $\phi(L_{5n}) \equiv 0 \pmod{10n}$.

(ii) $\phi(F_{5n}) \equiv 0 \pmod{80n}$; n is odd, $n > 1$.

Proof:

(i) The proof follows from (c) and (e) and the fact that $\phi(L_5) \equiv 0 \pmod{10}$ when $n = 1$.

(ii) Since, for odd n, $5n \equiv 5 \pmod{10}$ is true when $n > 1$, by (c) and (d) there exists a primitive prime divisor p of F_{5n} satisfying $p \equiv 1 \pmod{20n}$. Since 5 and p are both relatively prime factors of F_{5n}, $80n$ divides $\phi(F_{5n})$ by (a).

Theorem 5:

(i) If $k \geq 2$, then $\phi(F_{5^k}) \equiv 0 \pmod{16 \cdot 5^{2k-1}}$.

(ii) If $n = 2^{r+1} \cdot 3^m \cdot 5^k$ with $r \geq 1$, $m \geq 1$, $k \geq 1$, then $\phi(F_n) \equiv 0 \pmod{4n}$.

Proof:

(i) From [4], p. 192, we have $5^k | F_{5^k}$. Since $5^k \equiv 5 \pmod{10}$ is true, by (d) there exists a primitive prime divisor p satisfying $p \equiv 1 \pmod{4.5^k}$. As 5^k and p are relatively prime, $\phi(5^k) \cdot \phi(p)$ divides $\phi(F_k)$.

(ii) By [4], p. 192, we have $n | F_n$. This, along with Theorem 2, completes the proof.

A Final Note: It is desirable to shed some light on the cases not discussed thus far and on the difficulties encountered in the generalization process. This is done by showing that the following two congruences are not valid in general for a positive integer n.

(i) $\phi(L_{2n}) \equiv 0 \pmod{2n}$

(ii) $\phi(F_{2n+1}) \equiv 0 \pmod{2n + 1}$

In regard to (i), we observe that if, for a composite m, L_m is prime, then m must be of the form 2^t, where $t \geq 2$ (see [2]). Consequently, with $t \geq 2$ when L_{2^t} is prime, which is primitive, we have $\phi(L_{2^t}) = L_{2^t} - 1$.

As proved in Theorem 3, $L_{2^t} \equiv -1 \pmod{2^t}$; therefore, we can conclude that $\phi(L_{2^t}) \equiv -2 \pmod{2^t}$. Thus, it follows that $\phi(L_{2^t}) \not\equiv 0 \pmod{2^t}$ when L_{2^t} is prime and $t \geq 2$. Besides this, there may exist other Lucas numbers connected with this which may not satisfy the congruence of (i). One such illustration will be the members of the type $L_{2^t \cdot p}$, where p is an odd prime and $t \geq 1$. We observe that, for $n < 50$, $\phi(L_{2n}) \not\equiv 0 \pmod{2n}$ when n = 2, 4, 8, 11, 12, 17, 26, 29, 37, 46. In view of this, we conclude that the congruence relation in (i) is not true in general.

For case (ii), we observe that for odd n, if F_n is prime p, where $p \equiv \pm 3 \pmod{10}$, then $p \equiv (2n - 1) \pmod{4n}$ (see [1], p. 254).

Thus, under this hypothesis of primality, $\phi(F_n) \equiv -2 \pmod{n}$. It is easy to see that F_n is a prime of this type when n = 7, 13, 17, 23, 43, 47, 83. It is interesting to observe that if, for a prime subscript n, F_n is the product of two primitive primes each $\equiv \pm 3 \pmod{10}$, then $\phi(F_n) \equiv 4 \pmod{n}$. This is true when n = 59, 61, 71, 79, 101, 109.

Based on this, there may exist Fibonacci numbers of odd subscripts n, where n is composite, which may also not satisfy relation (ii). One such example is F_{161}, where $\phi(F_{161}) \equiv 16 \pmod{161}$. Consequently, we are justified to say that the congruence relation of (ii) is also not true in general.

References

1. J. Brillhart, P. L. Montgomery, & R. D. Silverman. "Tables of Fibonacci and Lucas Factorizations." *Math. Comp. 50.181* (1988):251-60.
2. J. L. Brown, Jr. Solution of Problem H-33. *Fibonacci Quarterly 2.4* (1964):309.
3. D. M. Burton. *Elementary Number Theory*. New York: Allyn and Bacon, 1980.
4. V. E. Hoggatt, Jr., & G. E. Bergum. "Divisibility and Congruence Relations." *Fibonacci Quarterly 12.2* (1974):189-95.
5. V. E. Hoggatt, Jr., & Hugh Edgar. "Another Proof that $\phi(F_n) \equiv 0 \pmod 4$ for all $n > 4$." *Fibonacci Quarterly 18.1* (1980):80.
6. D. Jarden. *Recurring Sequences*. 3rd ed. Jerusalem: Riveon Lematematika, 1973.
7. Sahib Singh. "Thoro's Conjecture and Allied Divisibility Property of Lucas Numbers." *Fibonacci Quarterly 18.2* (1980):135-37.

*****