PARTITIONS WITH "$M(a)$ COPIES OF a"

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In [1], Agarwal & Andrews studied partitions with "a copies of a," and in [2], Agarwal & Mullen studied partitions with "$d(a)$ copies of a" (where d is the divisor function). In this note, partitions with "$M(a)$ copies of a" are considered; the maximum exponent function, M, is defined by

$$M(a) = \max(e_1, \ldots, e_r)$$

if the integer $a > 1$ has canonical prime-power form $a = p_1^{e_1} \ldots p_r^{e_r}$, and $M(1) = 1$.

Define L to be the set of ordered pairs (a, b) of positive integers with $1 \le b \le M(a)$. We say π is a partition of n with $M(a)$ copies of a if π is a finite ordered collection $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ of elements of L such that $a_1 + a_2 + \cdots + a_k = n$ and, for $1 \le i \le j \le k$, $a_i \ge a_j$ with $b_i \le b_j$ if $a_i = a_j$. If we replace (a, b) in L by a_b, the partitions of n with "$M(a)$ copies of a" for $n = 1, 2, 3, 4$, can be represented, respectively, by

$$1_1;\ 2_1,\ 1_1 + 1_1;\ 3_1,\ 2_1 + 1_1,\ 1_1 + 1_1 + 1_1;$$

$$4_1,\ 4_2,\ 3_1 + 1_1,\ 2_1 + 2_1,\ 2_1 + 1_1 + 1_1,\ 1_1 + 1_1 + 1_1 + 1_1.$$

For the positive integer n, let $m(n)$ denote the number of partitions of n with "$M(a)$ copies of a." As in [3, Ch. 1] and [2], a generating function for such partitions is

$$1 + \sum_{n=1}^{\infty} m(n)q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-M(n)}.$$

This is an immediate consequence of the following theorem [3, Th. 1.1]:

If H is a set of positive integers, if "H" is the set of partitions with parts in H, and if $p(\text{"}H\text{"}, n)$ is the number of partitions of n with parts in H, then for $|q| < 1$,

$$\sum_{n \ge 0} p(\text{"}H\text{"}, n)q^n = \prod_{n \in H}^{\infty} (1 - q^n)^{-1}.$$

The factor $(1 - q^n)^{-1} = 1 + q^n + q^{n+n} + \cdots$ is replaced by

$$\begin{aligned}(1 - q^n)^{-M(n)} &= (1 + q^n + q^{n+n} + \ldots)^{M(n)} \\ &= (1 + q^{n_1} + q^{n_1 + n_1} + \ldots)(1 + q^{n_2} + q^{n_2 + n_2} + \ldots) \\ &\quad \cdots (1 + q^{n_{M(n)}} + q^{n_{M(n)} + n_{M(n)}} + \ldots)\end{aligned}$$

for $n_i = n$ $(1 \le i \le M(n))$; thus, the number of partitions of n with "$M(a)$ copies of a" is counted. For example, $m(4)$ is the coefficient of q^4 in

$$\begin{aligned}&(1 - q)^{-1}(1 - q^2)^{-1}(1 - q^3)^{-1}(1 - q^4)^{-2} \\ &= (1 + q^{1_1} + q^{1_1 + 1_1} + q^{1_1 + 1_1 + 1_1} + q^{1_1 + 1_1 + 1_1 + 1_1} + \ldots) \\ &\quad \cdot (1 + q^{2_1} + q^{2_1 + 2_1} + \ldots)(1 + q^{3_1} + \ldots)(1 + q^{4_1} + \ldots)(1 + q^{4_2} + \ldots)\end{aligned}$$

for $1_1 = 1$, $2_1 = 2$, $3_1 = 3$, $4_1 = 4_2 = 4$; since

$$q^4 = q^{4_1} = q^{4_2} = q^{3_1 + 1_1} = q^{2_1 + 2_1} = q^{2_1 + 1_1 + 1_1} = q^{1_1 + 1_1 + 1_1 + 1_1},$$

then $m(4) = 6$, and the exponents

$$4_1,\ 4_2,\ 3_1 + 1_1,\ 2_1 + 2_1,\ 2_1 + 1_1 + 1_1,\ 1_1 + 1_1 + 1_1 + 1_1$$

are the six partitions of 4 with "$M(a)$ copies of a."

If $p(n)$ is the number of unrestricted partitions of n, then

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} m(n)q^n &= \prod_{n=1}^{\infty}(1 - q^n)^{-1} \prod_{\substack{n>1 \\ M(n)>1}} (1 - q\ )^{-(M(n)-1)} \\ &= \left(\sum_{n=0}^{\infty} p(n)q^n\right) \prod_{\substack{n>1 \\ M(n)>1}} \left(\sum_{i=0}^{\infty} q^{in}\right)^{M(n)-1}. \end{aligned}$$

Note that $M(n) = p(n)$ if $n = 1, 2, 3$. Some values of $m(n)$ are shown below.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$m(n)$	1	2	3	6	8	13	18	30	41	60	82	121	162	226	302	422

A recurrence formula for $m(n)$ is now given. Let $[r]$ denote the greatest integer less than or equal to the real number r; let

$$\left(\sum_{i=0}^{\infty} q^i\right)^k = \sum_{v=0}^{\infty}(k)_v q^v,$$

where k is a positive integer [so that $(k)_v$ is the coefficient of q^v in the expanded form of $(1 + q + q^2 + q^3 + \cdots)^k$); and let s_i equal the i^{th} nonsquare-free positive integer (with $s_1 = 4$, $s_2 = 8$, $s_3 = 9$, $s_4 = 12$, $s_5 = 16$, and so forth). Then, if $n \geq 4$,

$$m(n) = \sum_{i=1}^{j} m_{n,s_i},$$

where j is the unique positive integer such that $s_j \leq n < s_{j+1}$, $m(0)$ is defined equal to 1,

$$m_{n,4} = p(n) + \sum_{i=1}^{[n/4]} p(4 - ni)$$

for $n \geq 4$, and

$$\begin{aligned} m_{n,s_j} = {} & (M(s_j) - 1)_{[n/s_j]}\ m(n - s_j[n/s_j]) \\ & + \sum_{i=1}^{[n/s_j]-1} (M(s_j) - 1)_i \left(\sum_{v=1}^{j-1} m_{n-s_j i,\, s_v}\right) \end{aligned}$$

for $n \geq s_j > s_1$. For example,

$$\begin{aligned} m(16) &= m_{16,16} + m_{16,12} + m_{16,9} + m_{16,8} + m_{16,4} \\ &= (M(16) - 1)_1 m(0) + (M(12) - 1)_1 m(4) + (M(9) - 1)_1 m(7) \\ &\quad + (M(8) - 1)_2 m(0) + (M(8) - 1)_1 m_{8,4} \\ &\quad + (p(16) + p(12) + p(8) + p(4) + p(0)) \\ &= (3)_1 \cdot 1 + (1)_1 \cdot 6 + (1)_1 \cdot 18 + (2)_2 \cdot 1 \\ &\quad + (2)_1(p(8) + p(4) + p(0)) + (231 + 77 + 22 + 5 + 1) \\ &= 3 + 6 + 18 + 3 + 56 + 336 \\ &= 422. \end{aligned}$$

Combinatorial interpretations of partitions with "$M(a)$ copies of a" can be stated in terms of plane partitions [3, Ch. 11] and factorization patterns [2]. In [4], Mitchell considers plane partitions in which the number of parts equal to $j \geq 1$ in any row is not less than the number of parts equal to j in the next row; we designate these plane partitions as Mitchell plane partitions (MPP's). Each MPP of a positive integer n can be written uniquely in an "identical-element-column format" (ICF) of the type

$$\begin{array}{cccc} a_{11} & a_{21} & \cdots & a_{r1} \\ \vdots & \vdots & & \vdots \\ a_{1t_1} & a_{2t_2} & \cdots & a_{rt_r} \end{array}$$

with

$$\sum_{i=1}^{r} \sum_{j=1}^{t_i} a_{ij} = n$$

and

$$a_{i1} \ge a_{i+1,1} \ (i = 1, \ldots, r-1),$$

with

$$a_{i1} = \cdots = a_{it_i}$$

for each $i = 1, \ldots, r$, and such that, if $a_{i1} = a_{k1}$ for $i < k$, then $t_i \ge t_k$. If $n > 1$, then $m(n)$ is the number of ICF's of the following types:

I. $a_{11} \quad a_{21} \quad \ldots \quad a_{r1}$ (with $\sum_{i=1}^{r} a_{i1} = n$ and $a_{i1} \ge a_{i+1,1}$ $(i = 1, \ldots, r-1)$; $t_1 = \cdots = t_r = 1$. ICF's of this type are unrestricted partitions of n.)

II. ICF's formed by first replacing one or more of any nonsquarefree a_{i1} $(i = 1, \ldots, r)$ in I, as indicated in (i) and (ii) below, and then rearranging these columns if necessary. (If a_{i1} is squarefree, then a_{i1} is the only acceptable form.)

(i) If $a_{i1} \ne a_{k1}$ for $k \ne i$, and p is the smallest prime such that $p^{M(a_{i1})}$ divides a_{i1}, then acceptable replacement forms for a_{i1} are those with a_{i1}/p^v identical column entries, each entry p^v $(v = 1, \ldots, M(a_i) - 1)$.

(ii) If $a_{i1} = a_{i+1,1} = \cdots = a_{i+w,1}$, $a_{i1} \ne a_{k1}$ if $k \ne i, i+1, \ldots, i+w$ $(1 \le i < i + r \le r)$, then acceptable replacements are those with one or more of $a_{i1}, \ldots, a_{i+w,1}$ replaced by replacement forms specified in (i) under the condition that entries in the column replacing a_{b1} are greater than or equal to entries in the column replacing a_{c1} if $c > b$ $(i \le b < c \le i + w)$.

Denote the set of ICF's of n of these types by MICF(n), and $m(n)$ is the order of the set MICF(n).

Also, $m(n)$ is the number of restricted "maximum-exponent" factorization patterns (MFP's) of the type $b_1^{a_1} \ldots b_r^{a_r}$ with

$$n = b_1 a_1 + \cdots + b_r a_r$$

and

$$b_1 = \cdots = b_{k_1} > b_{k_1+1} = \cdots = b_{k_2} > \cdots > b_{k_{c-1}+1} = \cdots = b_{k_c}$$

with

$$k_c = r$$

and

$$a_1 \ge \cdots \ge a_{k_1},\ a_{k_1+1} \ge \cdots \ge a_{k_2},\ \ldots,\ a_{k_{c-1}+1} \ge \cdots \ge a_{k_c},$$

and in which, for $b_v a_v = w$ $(1 \le w \le n)$ and for $v = 1, 2, \ldots, r$, $b_v^{a_v}$ has the following specified form:

(1) If w is a squarefree positive integer, then $b_v^{a_v} = w^1$;

(2) If w is not squarefree, and p is the smallest prime such that $p^{M(w)}$ divides w, then $b_v^{a_v} = w^1$ or $b_v^{a_v} = (p^t)^{(w/p^t)}$ $(t = 1, \ldots, M(n) - 1)$.

To illustrate, $m(8) = 30$ and the elements of MICF(8) are

8	2	4	71	62	611	53	521	5111	44	42	22	431	321	422	222
	2	4								2	22		2		2
	2														
	2														

4211 $\begin{array}{l}2211\\2\end{array}$ 41111 $\begin{array}{l}21111\\2\end{array}$ 332 3311 3221 32111 311111 2222 22211

221111 2111111 11111111

The element $\begin{array}{l}321\\\ 2\end{array}$ (obtained from $\begin{array}{l}231\\2\end{array}$ by a column rearrangement) has plane partition form $\begin{array}{l}321\\2\end{array}$ and corresponds to the MFP $3^1 2^2 1^1$.

For any positive integer n there is a bijection between the set MICF(n) and the set of MFP's of n. Also, a bijection between the members of the multiset $\{1_1, 2_1, 3_1, 4_1, 4_2, \ldots, n_1, \ldots, n_{M(n)}\}$ and the set of MFP's of n is indicated by 1_1 corresponding to 1^1, 2_1 to 2^1, 3_1 to 3^1, 4_1 to 2^2, 4_2 to 4^1, ..., and n_1 to n^1 if $M(n) = 1$, or n_1 to $p^{n/p}$, ..., $n_{M(n)-1}$ to $(p^{M(n)-1})^{(n/p^{M(n)-1})}$, $n_{M(n)}$ to n^1 if $M(n) > 1$ and p is the smallest prime such that $p^{M(n)}$ divides n.

References

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