EXTENSIONS OF CONGRUENCES OF GLAISHER AND NIELSEN CONCERNING STIRLING NUMBERS

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1. Introduction

Let $s(n, k)$ and $S(n, k)$ be the (unsigned) Stirling numbers of the first and second kinds, respectively. These numbers are well known and have been extensively studied; see, for example, [5, Ch. 5]. The generating functions are

(1.1) $$(-\log(1 - x))^k = k! \sum_{n=k}^{\infty} s(n, k)x^n/n!,$$

(1.2) $$(e^x - 1)^k = k! \sum_{n=k}^{\infty} S(n, k)x^n/n!.$$

Congruences for the Stirling numbers are apparently not well known. A few congruences for prime moduli can be found in [5, pp. 218-19, 229] and other books, but surprisingly little work has been done on this problem. Carlitz [4] worked out a method for finding congruences for $S(n, k) \pmod p$, where p is prime, and recent papers by the author [8], Kwong [10], Nijenhuis & Wilf [12], and Peele [14] indicate an increased interest in Stirling number congruences.

The main purpose of this paper is to extend results of Glaisher [6] and Nielsen [11, p. 338] by proving the following congruences: Let p be an odd prime, let n be a positive integer, and suppose $p^t \| n$; that is, p^t is the highest power of p dividing n. Let B_{2r} be the $2r^{\text{th}}$ Bernoulli number. For $0 < 2r < 2p - 2$ and $1 < 2r + 1 < 2p - 2$, we have

(1.3) $$s(n, n - 2r) \equiv \frac{-n}{2r}\binom{n-1}{2r}B_{2r} \pmod{p^{2t}},$$

(1.4) $$s(n, n - 2r - 1) \equiv -\frac{n^2(2r + 1)}{4r}\binom{n-1}{2r+1}B_{2r} \pmod{p^{3t}},$$

(1.5) $$S(n + 2r, n) \equiv \frac{n}{2r}\binom{n+2r}{2r}B_{2r} \pmod{p^{2t}},$$

(1.6) $$S(n + 2r + 1, n) \equiv \frac{n^2(2r + 1)}{4r}\binom{n+2r+1}{2r+1}B_{2r} \pmod{p^{3t}}.$$

When $n = p$ and $0 < 2r < p - 1$, $1 < 2r + 1 < p - 1$, congruences (1.3) and (1.4) reduce to the previously mentioned theorem of Glaisher, while (1.5) and (1.6) reduce to the results of Nielsen. Since extensive tables of the Bernoulli numbers are available (the first sixty are listed in [9, p. 234], for example) and, since the properties of the Bernoulli numbers are well known, perhaps congruences like (1.3)-(1.6) can give us information about the structure of the Stirling numbers. We note that applications of Glaisher's congruence are given in [2] and [6].

We also prove in this paper that, for $0 < m < 2p - 2$ and $p^t \| n$,

(1.7) $$s(n + m, n) \equiv \frac{n}{m}\binom{n+m}{m}(-1)^m B_m^{(m)} \pmod{p^{2t}},$$

(1.8) $$S(n, n - m) \equiv -\frac{n}{m}\binom{n-1}{m}B_m^{(m)} \pmod{p^{2t}},$$

where $B_m^{(m)}$ is a Bernoulli number of higher order. The numbers $B_m^{(m)}$ are discussed in [13, pp. 150-51, 461] and a table of the first thirteen values is given.

We shall actually prove (1.3)-(1.8) in a more general setting. Since it is just as easy to do so, we shall prove congruences for the *degenerate* Stirling numbers $s(n, k|\lambda)$ and $S(n, k|\lambda)$ of Carlitz [3]. By letting $\lambda = 0$, we obtain (1.3)-(1.8). We shall also show how to extend the range of r to $0 < 2r \le (p-1)p^t$ and $1 < 2r+1 < (p-1)p^t$, although the congruences become more complicated.

A summary by sections follows. Section 2 is a preliminary section in which we give the definitions and basic properties of the special numbers we need and state a theorem of Carlitz that is necessary for most of the results of this paper. In Section 3 we prove congruences (1.3)-(1.6) in terms of degenerate Stirling and Bernoulli numbers. In Section 4 we prove (1.7) and (1.8) in a more general setting. In Section 5 we extend (1.3)-(1.6) by increasing the range of r.

2. Preliminaries

The primary tool of this paper is the following theorem of Carlitz [2], who used it to prove the Glaisher and Nielsen congruences, as well as congruences for other special numbers.

Theorem 2.1 (Carlitz): Take

$$f = f(x) = \sum_{m=1}^{\infty} c_m x^m/m! \quad (c_1 = 1),$$

where the c_m are rational numbers and, for $k \ge 1$, define

$$\left(\frac{x}{f}\right)^k = \sum_{m=0}^{\infty} a_m^{(k)} x^m/m!,$$

with $a_m^{(1)} = a_m$. Define δ_m by means of

$$\frac{xf'}{f} = \sum_{m=0}^{\infty} \delta_m x^m/m!.$$

Then

$$(2.1) \qquad m a_m^{(k)} = -k \sum_{r=1}^{m} \binom{m}{r} \delta_r a_{m-r}^{(k)}.$$

Next we define and give properties of the degenerate Stirling numbers, the degenerate Bernoulli numbers, and other special numbers that we need.

Carlitz [3] defined the degenerate Stirling numbers of the first and second kinds, $s(n, k|\lambda)$ and $S(n, k|\lambda)$, by means of

$$(2.2) \qquad \left(\frac{1-(1-x)^\lambda}{\lambda}\right)^k = k! \sum_{n=k}^{\infty} s(n, k|\lambda) x^n/n!,$$

$$(2.3) \qquad ((1+\lambda x)^\mu - 1)^k = k! \sum_{n=k}^{\infty} S(n, k|\lambda) x^n/n!,$$

where $\lambda\mu = 1$. Comparing (2.2) and (2.3) with (1.1) and (1.2), we see that the limiting case $\lambda = 0$ gives the ordinary Stirling numbers. Carlitz [3] also defined $\beta_m^{(k)}(\lambda, z)$ by means of

$$(2.4) \qquad \left(\frac{x}{(1+\lambda x)^u - 1}\right)^k (1+\lambda x)^{\mu z} = \sum_{m=0}^{\infty} \beta_m^{(k)}(\lambda, z) x^m/m!$$

with the notation

$$(2.5) \qquad \beta_m^{(k)}(\lambda) = \beta_m^{(k)}(\lambda, 0).$$

We use the notation $\beta_m^{(1)}(\lambda, z) = \beta_m(\lambda, z)$. Thus,

$$(2.6) \quad \beta_m(\lambda,\ 0) = \beta_m(\lambda),$$

the degenerate Bernoulli number [1], and $\beta_m(0,\ 0) = B_m$, the ordinary Bernoulli number [5, p. 48]. It is known [3], that

$$(2.7) \quad s(k,\ k - m|\lambda) = (-1)^m\binom{k-1}{m}\beta_m^{(k)}(\lambda),$$

$$(2.8) \quad S(k + m,\ k|\lambda) = \binom{k+m}{m}\beta_m^{(-k)}(\lambda).$$

The author [7] defined $\alpha_m^{(k)}(\lambda)$ by means of

$$(2.9) \quad \left(\frac{\lambda x}{1 - (1 - x)^{\lambda}}\right)^k = \sum_{m=0}^{\infty}\alpha_m^{(k)}(\lambda)x^m/m!,$$

and showed that

$$(2.10) \quad s(k + m,\ k|\lambda) = \binom{k+m}{k}\alpha_m^{(-k)}(\lambda),$$

$$(2.11) \quad S(k,\ k - m|\lambda) = (-1)^m\binom{k-1}{m}\alpha_m^{(k)}(\lambda).$$

We shall make use of the numbers $\beta_m(\lambda,\ 1 - \lambda)$. It follows from (2.4) that, when $\lambda = 0$, we have

$$\beta_m(\lambda,\ 1 - \lambda) = \beta_m(0,\ 1) = \begin{cases} B_m & \text{when } m > 1, \\ -B_1 & \text{when } m = 1. \end{cases}$$

Also, from (2.4), we see that $\beta_0(\lambda,\ 1 - \lambda) = 1$, $\beta_1(\lambda,\ 1 - \lambda) = (1 - \lambda)/2$, and, for $m > 1$,

$$(2.12) \quad \beta_m(\lambda,\ 1 - \lambda) = \beta_m(\lambda) - m\lambda\beta_{m-1}(\lambda).$$

It follows from (2.12), by induction on m, that $\beta_m(\lambda,\ 1 - \lambda)$ satisfies a degenerate Staudt-Clausen theorem in exactly the same way that $\beta_m(\lambda)$ does [1]. Thus, we can say that, if p is a prime number and if λ is rational, $\lambda = a/b$ with b not divisible by p, then, for $r > 0$,

$$(2.13) \quad p\beta_{2r}(\lambda,\ 1 - \lambda) \equiv \begin{cases} -1 \pmod{p} & \text{if } (p - 1)|2r \text{ and } p|a, \\ 0 \pmod{p} & \text{otherwise.} \end{cases}$$

$$(2.14) \quad 2p\beta_{2r+1}(\lambda,\ 1 - \lambda) \equiv 0 \pmod{p}.$$

Note that, if λ is integral (mod p) and $\lambda \not\equiv 0 \pmod{p}$, then $\beta_m(\lambda,\ 1 - \lambda)$ is integral (mod p). It follows that, if λ is integral (mod p), then

$$m\lambda\beta_{m-1}(\lambda,\ 1 - \lambda) \equiv 0 \pmod{m}.$$

Now suppose p is an odd prime and $m \equiv 0 \pmod{p^w}$. It follows from (2.12) and properties of $\beta_m(\lambda)$ [1] that, if $m \not\equiv 0 \pmod{p - 1}$ and/or $a \not\equiv 0 \pmod{p}$, then

$$(2.15) \quad \beta_m(\lambda,\ 1 - \lambda) \equiv 0 \pmod{p^w}.$$

3. Extensions of the Glaisher-Nielsen Results

In this section, and in Sections 4 and 5, we always assume that p is an odd prime, n is a positive integer, and $p^t \| n$. We also assume λ is rational and integral (mod p); that is, $\lambda = a/b$ with b not divisible by p.

If we apply Theorem 2.1 with $f(x) = (1 + \lambda x)^{\mu} - 1$, and $\lambda\mu = 1$, we see that

$$a_m^{(k)} = \beta_m^{(k)}(\lambda), \quad \delta_m = \beta_m(\lambda,\ 1 - \lambda) \quad (m \geq 1),$$

where $\beta_m^{(k)}(\lambda)$ is defined by (2.4) and (2.5) and $\beta_m(\lambda,\ 1 - \lambda)$ is defined by (2.4) with $k = 1$. Thus, (2.1) becomes

$$(3.1)\qquad \beta_m^{(k)}(\lambda) = -\frac{k}{m}\sum_{r=1}^{m}\binom{m}{r}\beta_r(\lambda,\ 1-\lambda)\beta_{m-r}^{(k)}(\lambda),\ \text{with } \beta_0^{(k)}(\lambda) = 1.$$

Note that $\beta_m^{(n)}(\lambda)$ and $\beta_m^{(-n)}(\lambda)$ are integral (mod p) for $m < (p-1)p^t$.

Theorem 3.1: For $m = 1, \ldots, 2p-3$,

$$-\beta_m^{(-n)}(\lambda) \equiv \beta_m^{(n)}(\lambda) \equiv -\frac{n}{m}\,\beta_m(\lambda,\ 1-\lambda) \pmod{p^{2t}}.$$

If $\lambda \not\equiv 0 \pmod p$, the congruence is valid for $m = 1, \ldots, (p-1)p^t$.

Proof: From (3.1) and the properties (2.13)-(2.15) of $\beta_r(\lambda,\ 1-\lambda)$, we see that

$$(3.2)\qquad \beta_m^{(-n)}(\lambda) \equiv \beta_m^{(n)}(\lambda) \equiv 0 \pmod{p^t},$$

for $m = 1, 2, \ldots, 2p-3$, $m \neq p-1$, if $\lambda \equiv 0 \pmod p$. If $m = p-1$, then

$$\beta_{p-1}^{(-n)}(\lambda) \equiv \beta_{p-1}^{(n)}(\lambda) \equiv 0 \pmod{p^{t-1}}.$$

If $\lambda \not\equiv 0 \pmod p$, congruence (3.2) holds for $m = 1, \ldots, (p-1)p^t$. We note that $\binom{m}{p-1} \equiv 0 \pmod p$ for $m = 1, 2, \ldots, 2p-3$, $m \neq p-1$. Thus, letting $k = n$ or $k = -n$, we see that every term on the right side of (3.1), with the exception of of the $r = m$ term, is divisible by p^{2t}. This completes the proof.

The following corollary is immediate from (2.7) and (2.8).

Corollary 3.1: For $m = 1, \ldots, 2p-3$,

$$s(n,\ n-m|\lambda) \equiv (-1)^{m-1}\frac{n}{m}\binom{n-1}{m}\beta_m(\lambda,\ 1-\lambda) \pmod{p^{2t}},$$

$$S(n+m,\ n|\lambda) \equiv \frac{n}{m}\binom{n+m}{m}\beta_m(\lambda,\ 1-\lambda) \pmod{p^{2t}}.$$

If $\lambda \not\equiv 0 \pmod p$, the congruences are valid for $m = 1, 2, \ldots, (p-1)p^t$.

If m is even and we let $\lambda = 0$ in Corollary 3.1, we obtain congruences (1.3) and (1.5). If m is odd and $m > 1$, we see from Theorem 3.1 that

$$B_m^{(-n)} \equiv B_m^{(n)} \equiv 0 \pmod{p^{2t}},$$

where $B_m^{(k)}$ is the Bernoulli number of order k, defined by (2.5) with $\lambda = 0$. This is true because the Bernoulli number B_m is 0 when m is odd, $m > 1$. Thus, when $\lambda = 0$, each term on the right side of (3.1), with the exception of the $r = 1$ and $r = m-1$ terms, is divisible by p^{3t}. Hence,

$$\begin{aligned} B_{2r+1}^{(n)} &\equiv \frac{-n}{2r+1}\left[-(2r+1)B_1B_{2r}^{(n)} + (2r+1)B_{2r}B_1^{(n)}\right] \\ &\equiv \frac{n}{2r+1}\left[\frac{(2r+1)}{2}\cdot\frac{n}{2r}\,B_{2r} + (2r+1)\frac{n}{2}\,B_{2r}\right] \\ &\equiv \frac{n^2(2r+1)}{4r}\,B_{2r} \pmod{p^{3t}}. \end{aligned}$$

Similarly,

$$B_{2r+1}^{(-n)} \equiv \frac{n^2(2r+1)}{4r}\,B_{2r} \pmod{p^{3t}}.$$

Thus, we can state the following corollary.

Corollary 3.2: The ordinary Stirling numbers satisfy congruences (1.3)-(1.6).

It is not difficult to extend the range of m in Theorem 3.1. We do this in Section 5.

4. Further Congruences for the Stirling Numbers

In this section we prove congruences (1.7) and (1.8). Throughout the section, we still have the assumptions concerning n, p, t, and λ stated at the beginning of Section 3.

We first apply Theorem 2.1 to

$$f(x) = \frac{1 - (1 - x)^{\lambda}}{\lambda}.$$

We define $\alpha_m^{(k)}(\lambda)$ by (2.9), and we define $A_m(\lambda)$ by means of

$$(4.1) \qquad \frac{x}{f} \cdot f' = \frac{x\lambda}{[1 - (1 - x)^{\lambda}][(1 - x)^{1-\lambda}]} = \sum_{m=0}^{\infty} A_m(\lambda)x^m/m!$$

Then we have, by Theorem 2.1,

$$(4.2) \qquad \alpha_m^{(k)}(\lambda) = -\frac{k}{m}\sum_{r=1}^{m}\binom{m}{r}A_r(\lambda)\alpha_{m-r}^{(k)}(\lambda).$$

When $\lambda = 0$, (4.1) reduces to

$$\frac{x}{(1 - x)\ell n(1 - x)} = \sum_{m=0}^{\infty}(-1)^m B_m^{(m)}x^m/m!,$$

where $B_m^{(m)}$ is the Bernoulli number of higher order [10, pp. 150-51].

Lemma 4.1: If $A_m(\lambda)$ is defined by (4.1) and $\beta_m^{(k)}(\lambda)$ by (2.4) and (2.5), then

$$A_m(\lambda) = (-1)^m\beta_m^{(m)}(\lambda) \quad (m = 0, 1, 2, \ldots).$$

Proof: We first note that $\beta_m^{(m)}(0) = B_m^{(m)}$.

Using (2.5) and (2.6), we can prove by induction on z that, for all positive integers z,

$$(4.3) \qquad \beta_m^{(m)}(\lambda, z) = \sum_{j=0}^{m}\binom{m}{j}\binom{z}{j}j!\,\beta_{m-j}^{(m-j)}(\lambda).$$

Equation (4.3) is valid for all real z since $B_m^{(m)}(\lambda, z)$ is a polynomial in z. We note that a more general result could be proved for numbers generated by $(x/f)^k(f + 1)^z$, with f defined by Theorem 2.1. From (2.4), we also have

$$(4.4) \qquad \beta_m^{(m)}(\lambda, \lambda) - \beta_m^{(m)}(\lambda) = m\lambda\beta_{m-1}^{(m)}(\lambda) = m\lambda(\lambda - 1)\cdots(\lambda - m + 1).$$

Simplifying (4.3), with the aid of (4.4), we have, for $\lambda \neq 0$,

$$(4.5) \qquad \sum_{j=0}^{m}\frac{(\lambda - 1)(\lambda - 2)\cdots(\lambda - m + j)}{(m - j + 1)!j!}\beta_j^{(j)}(\lambda) = \frac{(\lambda - 1)(\lambda - 2)\cdots(\lambda - m)}{m!},$$

with $\beta_0^{(0)}(\lambda) = 1$. By means of (4.1) we can show that $(-1)^j A_j(\lambda)$ satisfies the same recurrence with $A_0(\lambda) = 1$. This completes the proof.

Thus, we can write, for all λ,

$$(4.6) \qquad \alpha_m^{(k)}(\lambda) = \frac{-k}{m}\sum_{r=1}^{m}\binom{m}{r}(-1)^r\beta_r^{(r)}(\lambda)\alpha_{m-r}^{(k)}(\lambda).$$

Before proving the main result of this section, we need to examine the properties of $\beta_r^{(r)}(\lambda)$. The first few values are given in the following table.

r	0	1	2	3
$\beta_r^{(r)}(\lambda)$	1	$(\lambda - 1)/2$	$(\lambda - 1)(\lambda - 5)/6$	$-3(\lambda - 1)(\lambda - 3)/4$

For $\lambda = 0$, the first thirteen values are given in [11, p. 461]. By the recurrence (4.5), we see that if p is an odd prime,

$$p\beta_m^{(m)}(\lambda) \equiv 0 \pmod{p} \quad (m = 1, \ldots, p-2),$$

$$p\beta_{p-1}^{(p-1)}(\lambda) \equiv \begin{cases} 0 \pmod{p} & \text{if } \lambda \not\equiv 0 \pmod{p}, \\ 1 \pmod{p} & \text{if } \lambda \equiv 0 \pmod{p}, \end{cases}$$

$$\beta_p^{(p)}(\lambda) \equiv 0 \pmod{p},$$

$$p\beta_{p+k}^{(p+k)}(\lambda) \equiv 0 \pmod{p} \quad (k = 1, \ldots, p-3)$$

$$p\beta_{2p-2}^{(2p-2)}(\lambda) \equiv \begin{cases} 0 \pmod{p} & \text{if } \lambda \not\equiv 0 \pmod{p} \\ -1 \pmod{p} & \text{if } \lambda \equiv 0 \pmod{p}, \end{cases}$$

$$p\beta_{2p-1}^{(2p-1)}(\lambda) \equiv \begin{cases} 0 \pmod{p} & \text{if } \lambda \not\equiv 0 \pmod{p} \\ 1/2 \pmod{p} & \text{if } \lambda \equiv 0 \pmod{p}, \end{cases}$$

$$\beta_{2p}^{(2p)}(\lambda) \equiv 0 \pmod{p} \quad (p > 3).$$

Theorem 4.1: For $p > 2$ and $m = 1, \ldots, 2p - 3$,

$$\alpha_m^{(n)}(\lambda) \equiv -\alpha_m^{(-n)}(\lambda) \equiv (-1)^{m+1}\frac{n}{m}\beta_m^{(m)}(\lambda) \pmod{p^{2t}}.$$

Proof: From (4.6) and properties of $\beta_r^{(r)}(\lambda)$, we see that, for $1 \le m \le 2p - 3$, $m \ne p - 1$,

$$\alpha_m^{(-n)}(\lambda) \equiv \alpha_m^{(n)}(\lambda) \equiv 0 \pmod{p^t}.$$

Also

$$\alpha_{p-1}^{(n)}(\lambda) \equiv 0 \pmod{p^{t-1}}.$$

Since $\binom{m}{p-1} \equiv 0 \pmod{p}$ for $1 \le m \le 2p - 3$, $m \ne p - 1$, we see that every term on the right side of (4.6) (when $k = n$ or $k = -n$) is divisible by p^{2t}, except the $r = m$ term. This completes the proof.

The next corollary follows immediately from (2.10) and (2.11).

Corollary 4.1: For $m = 1, \ldots, 2p - 3$,

$$s(n+m, n|\lambda) \equiv \frac{n}{m}\binom{n+m}{n}(-1)^m\beta_m^{(m)}(\lambda) \pmod{p^{2t}},$$

$$S(n, n-m|\lambda) \equiv -\frac{n}{m}\binom{n-1}{m}\beta_m^{(m)}(\lambda) \pmod{p^{2t}}.$$

Corollary 4.2: The ordinary Stirling numbers satisfy congruences (1.7) and (1.8).

5. Extensions of Congruences (1.3)-(1.6)

Let n, p, t, and λ be defined as in Section 3. Suppose m and h are such that $2p - 2 \le m$ and

$$(p-1)p^{h-1} < m \le (p-1)p^h < (p-1)p^t.$$

Then we define $f(t, h)$ by

$$f(t, h) = \begin{cases} 2t - h & \text{if } m \not\equiv 0 \pmod{p-1}, \\ 2t - 1 & \text{if } m \equiv 0 \pmod{p-1},\ m \not\equiv 0 \pmod{p},\ h = 1, \\ 2t - h - 1 & \text{if } m \equiv 0 \pmod{p-1},\ m \not\equiv 0 \pmod{p},\ h > 1, \\ 2t - h - u - 1 & \text{if } m \equiv 0 \pmod{p(p-1)},\ p^u \| m. \end{cases}$$

We now extend Theorem 3.1.

Theorem 5.1: Suppose $\lambda \equiv 0 \pmod{p}$. With m, h, and $f(t, h)$ defined as above, we have

$$-\beta_m^{(-n)}(\lambda) \equiv \beta_m^{(n)}(\lambda) \equiv -\frac{n}{m}\beta_m(\lambda, 1-\lambda) \pmod{p^{f(t,h)}}.$$

Proof: We first note that Theorem 5.1 implies that, if m is restricted as in the statement of the theorem, then

$$(5.1)\qquad \beta_m^{(-n)}(\lambda) \equiv \beta_m^{(n)}(\lambda) \equiv 0 \begin{cases} (\text{mod } p^t) \text{ if } m \not\equiv 0 \ (\text{mod } p-1), \\ (\text{mod } p^{t-1}) \text{ if } m \equiv 0 \ (\text{mod } p-1),\ m \not\equiv 0 \ (\text{mod } p), \\ (\text{mod } p^{t-u-1}) \text{ if } m \equiv 0 \ (\text{mod } p(p-1)),\ p^u \| m. \end{cases}$$

We first look at the case $m = 2p - 2$. In (3.1), with $k = n$ or $k = -n$, and $m = 2p - 2$, all terms on the right side with $r < m$ are divisible by p^{2t} except the term $r = p - 1$. We have

$$\frac{n}{2p-2}\binom{2p-2}{p-1}\beta_{p-1}(\lambda,\ 1-\lambda)\beta_{p-1}^{(\pm n)}(\lambda) \equiv 0 \ (\text{mod } p^{2t-1}),$$

so

$$-\beta_{2p-2}^{(-n)}(\lambda) \equiv \beta_{2p-2}^{(n)}(\lambda) \equiv -\frac{n}{2p-2}\,\beta_{2p-2}(\lambda,\ 1-\lambda) \ (\text{mod } p^{2t-1}).$$

We now use induction on m. Assume Theorem 5.1 is true for all positive integers r such that $2p - 2 \le r < m$. In particular, assume congruences (5.1) hold with m replaced by r. The problem is to show in (3.1) that, for $r = 1, \ldots, m - 1$,

$$(5.2)\qquad \frac{n}{m}\binom{m}{r}\beta_r(\lambda,\ 1-\lambda)\beta_{m-r}^{(\pm n)}(\lambda) \equiv 0 \ (\text{mod } p^{f(t,h)}).$$

By using the induction hypothesis and the properties of $\beta_r(\lambda,\ 1-\lambda)$ discussed in Section 2, we can routinely show that (5.2) holds for all cases of m given in the definition of $f(t, h)$.

Corollary 5.1: With the hypotheses of Theorem 5.1,

$$s(n,\ n-m|\lambda) \equiv \frac{n}{m}(-1)^{m-1}\binom{n-1}{m}\beta_m(\lambda,\ 1-\lambda) \ (\text{mod } p^{f(t,h)}),$$

$$S(n+m,\ n|\lambda) \equiv \frac{n}{m}\binom{n+m}{m}\beta_m(\lambda,\ 1-\lambda) \ (\text{mod } p^{f(t,h)}).$$

Now let

$$g(t,h) = \begin{cases} 3t - h - w - 1 \text{ if } 2r \equiv 0 \ (\text{mod } p(p-1)),\ p^w \| 2r, \\ 3t - h - u - 1 \text{ if } p^u \| (2r+1),\ u \ge 1, \\ 3t - h - 1 \text{ in all other cases.} \end{cases}$$

By letting $\lambda = 0$ in Corollary 5.1, we can now prove the following extensions of (1.3)-(1.6).

Corollary 5.2: Let $2r$ and $2r + 1$ be restricted as m is restricted in Theorem 5.1. Then

$$s(n,\ n-2r) \equiv \frac{-n}{2r}\binom{n-1}{2r}B_{2r} \ (\text{mod } p^{f(t,h)}),$$

$$S(n+2r,\ n) \equiv \frac{n}{2r}\binom{n+2r}{2r}B_{2r} (\text{mod } p^{f(t,h)}),$$

$$s(n,\ n-2r-1) \equiv \frac{-n^2(2r+1)}{4r}\binom{n-1}{2r+1}B_{2r} \ (\text{mod } p^{g(t,h)}),$$

$$S(n+2r+1,\ h) \equiv \frac{n^2(2r+1)}{4r}\binom{n+2r+1}{2r+1}B_{2r} \ (\text{mod } p^{g(t,h)}).$$

If $m > (p-1)p^{t-1}$, the congruences become more complicated. However, using the same kind of reasoning as before, we can state the following result. We let $f(t, t)$ be defined as in Theorem 5.1 and define y_1 and y_2 by

$$p^{y_1}\left\|\binom{n-1}{m}\right.,\quad p^{y_2}\left\|\binom{n+m}{m}\right..$$

Theorem 5.2: Let $(p - 1)p^{t-1} < m \le (p - 1)p^t$ and $m \not\equiv 0 \pmod{(p - 1)p^{t-1}}$. Then

$$s(n, n - m|\lambda) \equiv \frac{n}{m}(-1)^{m-1}\binom{n-1}{m}\beta_m(\lambda, 1 - \lambda) \pmod{p^{y_1 + f(t, t)}},$$

$$S(n + m, n|\lambda) \equiv \frac{n}{m}\binom{n+m}{m}\beta_m(\lambda, 1 - \lambda) \pmod{p^{y_2 + f(t, t)}}.$$

By letting $\lambda = 0$ in Theorem 5.2, we get the corresponding congruences for the ordinary Stirling numbers.

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