THE DETERMINATION OF A CLASS OF PRIMITIVE INTEGRAL TRIANGLES

Joseph E. Carroll and Ken Yanosko
Humboldt State University, Arcata, CA 95521
(Submitted December 1988)

One of the problems of classical number theory is the determination of all primitive integral right triangles. The well-known answer is that if $r > s$ are relatively prime positive integers, not both odd, then the triangle with sides $r^2 - s^2$, $2rs$, and $r^2 + s^2$ is such a triangle (easy to check) and any such triangle is of this form for some r and s. A simple proof of the latter half is given in [1]. This paper deals with a similar question that has a similar answer but a somewhat longer solution. The main tool in that solution is a thinly disguised version of the Chebyshev polynomials of the second kind.

Definition 1: Let $j \geq k$ be positive, relatively prime integers. A triangle will be called an $\langle j, k\rangle$ triangle if one of its angles is j/k times another.

It is easy to write down the primitive integral $\langle 1, 1\rangle$ (i.e., isosceles) triangles. These triangles have sides s, s, and r, where r and s are positive integers, $(r, s) = 1$, and $r < 2s$. The primitive integral $\langle 2, 1\rangle$ triangles have been determined by Luthar in [2]. If r and s are positive integers where $(r, s) = 1$ and $s < r < 2s$, then the triangle with sides rs, s^2, and $r^2 - s^2$ is a primitive integral $\langle 2, 1\rangle$ triangle, and all such triangles are of this form for suitable r and s. In this paper we shall determine all primitive integral $\langle j, k\rangle$ triangles for all j and k satisfying the criterion of Definition 1. Although this is hardly one of the burning mathematical questions of our time, it is hoped that the solution presented here will be of some interest, since it both draws ideas from several areas of mathematics and requires little background to understand.

First, let us fix j and k. It is clear that the $\langle j, k\rangle$ triangles are characterized by having angles $j\alpha$, $k\alpha$, and $\pi - (j + k)\alpha$ for some positive real number α such that $(j + k)\alpha < \pi$. Also, for any such α, there may or may not be a rational sided (hence, a primitive integral) triangle in the similarity class of $\langle j, k\rangle$ triangles associated with α in this way. The law of sines immediately gives us a triangle in that similarity class. If the triangle with sides a, b, c is denoted by the triple $\langle a, b, c\rangle$, then $\langle \sin j\alpha, \sin k\alpha, \sin(j + k)\alpha\rangle$ is in it. The following lemma leads us to a condition on α sufficient to ensure that there is a rational sided triangle similar to $\langle \sin j\alpha, \sin k\alpha, \sin(j + k)\alpha\rangle$.

Lemma 1: Define a sequence $\{p_n(x)\}_{n\geq 0}$ of polynomials with integer coefficients as follows: $p_0(x) \equiv 0$, $p_1(x) \equiv 1$, and, for $n \geq 2$,

$$p_n(x) = xp_{n-1}(x) - p_{n-2}(x).$$

Then, for any real number α which is not an integral multiple of π, we have

$$p_n(2\cos\alpha) = (\sin n\alpha)/(\sin\alpha).$$

Proof: The formula for the sine of a sum yields the following identities for $n \geq 2$:

$$\sin n\alpha = \sin(n - 1)\alpha\cos\alpha + \cos(n - 1)\alpha\sin\alpha$$

$$\sin(n - 2)\alpha = \sin(n - 1)\alpha\cos\alpha - \cos(n - 1)\alpha\sin\alpha$$

Adding these identities and dividing by $\sin\alpha$, we get:

$$(\sin n\alpha)/(\sin\alpha) = (2\cos\alpha)\cdot(\sin(n - 1)\alpha)/(\sin\alpha) - (\sin(n - 2)\alpha)/(\sin\alpha)$$

Thus, for any α which is not an integral multiple of π, the sequences

$$\{(\sin n\alpha)/(\sin \alpha)\}_{n \ge 0} \quad \text{and} \quad \{p_n(2 \cos \alpha)\}_{n \ge 0}$$

satisfy the same second-order linear recurrence relation. Furthermore, these sequences coincide on their first two terms. It follows that they are identical for all n.

Proposition 1: If $0 < \alpha < \pi/(j+k)$ and $\cos \alpha$ is a rational number, then there is a rational sided triangle with angles $j\alpha$, $k\alpha$, and $\pi - (j+k)\alpha$.

Proof: By Lemma 1, $\langle p_j(2 \cos \alpha), p_k(2 \cos \alpha), p_{j+k}(2 \cos \alpha)\rangle$ has the correct angles. Its sides are rational because $\cos \alpha$ is.

Remark 1: It is clear from the definition of $\{p_n\}$ that, for all $n \ge 1$, $p_n(x)$ is monic of degree $n - 1$. These polynomials, after a shift of subscripts and a change of variables, are none other than the Chebyshev polynomials of the second kind, $\{U_n(x)\}_{n \ge 0}$. For $n \ge 0$, $U_n(x) = p_{n+1}(2x)$. In fact, Lemma 1 is equivalent to a well-known property of U_n. It is proved again here to keep the discussion self-contained. The Chebyshev polynomials of the first kind, $\{T_n(x)\}_{n \ge 0}$, also deserve mention because they are used in the proof of the following lemma, which will lead us to the converse of Proposition 1. They can be defined by

$$T_0(x) \equiv 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \text{ for } n \ge 2.$$

Reasoning as in the proof of Lemma 1, one can show that, for any real number α, $T_n(\cos \alpha) = \cos n\alpha$.

Lemma 2: Let σ, τ be real numbers; then, for any integers m and n, $\cos(m\sigma + n\tau)$ is in the $\mathbf{Z}[\cos \sigma, \cos \tau]$ module generated by 1 and $\cos(\sigma + \tau)$.

Proof: Suppose that $m, n \ge 0$. Then

$$\begin{aligned}\cos(\pm(m\sigma \pm n\tau)) &= \cos m\sigma \cos n\tau \mp \sin m\sigma \sin n\tau \\ &= T_m(\cos \sigma)T_n(\cos \tau) \\ &\qquad \mp \sin \sigma\, p_m(2 \cos \sigma) \sin \tau\, p_n(2 \cos \tau).\end{aligned}$$

This follows from Lemma 1 and Remark 1 and is also true if σ or τ is an integral multiple of π. Using the formula for the cosine of a sum again, we deduce

$$\begin{aligned}\cos(\pm(m\sigma \pm n\tau)) &= T_m(\cos \sigma)T_n(\cos \tau) \\ &\qquad \pm p_m(2 \cos \sigma)p_n(2 \cos \tau)(\cos(\sigma + \tau) - \cos \sigma \cos \tau).\end{aligned}$$

Proposition 2: Suppose that for positive relatively prime integers $j \ge k$ with $0 < \alpha < \pi/(j+k)$ there is a rational sided triangle with angles $j\alpha$, $k\alpha$, and $\pi - (j+k)\alpha$. Then $\cos \alpha$ is a rational number.

Proof: If such a rational $\langle j, k\rangle$ triangle exists, then the law of cosines tells us that $\cos j\alpha$, $\cos k\alpha$, and $\cos(j+k)\alpha = -\cos(\pi - (j+k)\alpha)$ are all rational. Since j and k are relatively prime, there are integers m and n such that $mj + nk = 1$. Applying Lemma 2 for $\sigma = j\alpha$ and $\tau = k\alpha$, and using this m and n, we deduce that $\cos \alpha$ is rational, as claimed.

We now have necessary and sufficient conditions on α that there be a rational sided triangle with angles $j\alpha$, $k\alpha$, and $\pi - (j+k)\alpha$. When there is such a triangle, we need to find the primitive integral triangle in its similarity class. Properties of the sequence $\{p_n(x)\}$ and of a related sequence of homogeneous polynomials are the tools that will allow us to make that determination.

Proposition 3: The following are true for the sequence $\{p_n(x)\}$ defined in the statement of Lemma 1:

(a) $p_n(x) = \sum_{i=0}^{[(n-1)/2]} (-1)^i \binom{n-1-i}{i} x^{n-1-2i}$, for $n \geq 0$;

(b) $p_n(x) = \prod_{t=1}^{n-1} (x - 2\cos(t\pi/n))$, for $n \geq 1$;

(c) If $d|n$, then $p_d(x)|p_n(x)$ as polynomials in $\mathbf{Z}[x]$.

Proof: (a) A straightforward (if somewhat tedious) computation using a standard addition formula for binomial coefficients demonstrates that the sequence of candidate polynomials shown above satisfies the defining recurrence relation for the p_n. It is immediate that the two sequences coincide for $n = 0, 1$, so they must be the same for all n. Like Proposition 1, this is equivalent to a well-known statement about the U_n.

(b) Lemma 1 implies that $2\cos(t\pi/n)$ is a root of p_n for $t = 1, 2, \ldots, n - 1$ and, since the cosine is strictly decreasing on $[0, \pi]$, these roots are distinct. Since p_n has degree $n - 1$, the proposed equation is true up to multiplication by a constant. But, both p_n and the product above are monic, so the constant is 1.

(c) Part (b) implies this divisibility property as polynomials over the real numbers. If $p_n(x) = p_d(x)q(x)$, where $q(x)$ has real coefficients, the fact that p_d is integer monic and p_n is integral implies that q is integral. In fact, extending this reasoning, one can prove a stronger statement: if m and n are nonnegative integers, then $p_{(m,n)}(x)$ is the greatest common divisor of $p_m(x)$ and $p_n(x)$ in $\mathbf{Z}[x]$.

Remark 2: The field extension $\mathbf{Q}(e^{2\pi i/q})/\mathbf{Q}$ for q an odd prime is often used as an example in the teaching of Galois theory and algebraic number theory. It is shown that this extension is Galois of degree $q - 1$ with cyclic Galois group and that the irreducible polynomial of $e^{2\pi i/q}$ over $\mathbf{Q}$ is

$$\Phi_q(x) = x^{q-1} + \cdots + 1.$$

It is also shown that the unique subextension of index 2, which is the subfield fixed by complex conjugation, is generated by

$$2\cos(2\pi/q) = e^{2\pi i/q} + e^{-2\pi i/q},$$

an algebraic integer. Using Proposition 3(b), an identity satisfied by the $\{p_n\}$ that is easily proved, and some basic Galois theory, it can be shown that the irreducible polynomial of $2\cos(2\pi/q)$ over $\mathbf{Q}$ is

$$p_{(q+1)/2}(x) + p_{(q-1)/2}(x).$$

Proposition 3(a) then yields an explicit expression.

It is convenient to introduce a new sequence $\{P_n(x, y)\}_{n \geq 1}$ of homogeneous polynomials associated to $\{p_n(x)\}$. For $n \geq 1$, let

$$P_n(x, y) = y^{n-1}p_n(x/y) = \sum_{i=0}^{[(n-1)/2]} (-1)^i \binom{n-1-i}{i} x^{n-1-2i}y^{2i},$$

where the latter equation above follows from Proposition 3(a). Using Proposition 3(c), we immediately see that $d|n$ implies $P_d|P_n$ as polynomials in $\mathbf{Z}[x, y]$. We require a final lemma before stating and proving the main result of this paper.

Lemma 3: Let r and s be positive integers with $(r, s) = 1$ and let $n \geq 1$. Then

(a) $(s, P_n(r, s)) = 1$;

(b) $(P_n(r, s), P_{n+1}(r, s)) = 1$.

Proof: (a) First, we observe that $P_n(r, s) \equiv r^{n-1} \pmod{s}$. This follows either from the explicit expression for P_n given above or directly from the definition of P_n and the fact, noted in Remark 1, that p_n is integral monic of degree $n - 1$. Since $(r, s) = 1$, it follows that $(s, P_n(r, s)) = 1$ for $n \geq 1$.

(b) We prove this part by induction. Since $P_1(r, s) = 1$, the statement is true for $n = 1$. Let $n \geq 2$ and assume that the statement is true for $n - 1$. By the definition of the sequence $\{P_n(x, y)\}$, the defining recursion formula for $\{p_n(x)\}$ translates to

$$P_{n+1}(r, s) = rP_n(r, s) - s^2P_{n-1}(r, s).$$

Assume d is a positive integer such that $d|P_n(r, s)$ and $d|P_{n+1}(r, s)$. Then, by part (a), $(d, s) = 1$; by the equation above, $d|s^2P_{n-1}(r, s)$; thus $d|P_{n-1}(r, s)$. Therefore, by the induction assumption, $d = 1$.

Theorem 1: Let $j \geq k$ be positive integers with $(j, k) = 1$, and let r and s be positive integers with $(r, s) = 1$ and $\cos(\pi/(j + k)) < r/2s < 1$. Then

$$\langle s^kP_j(r, s),\ s^jP_k(r, s),\ P_{j+k}(r, s)\rangle$$

is a primitive integral $\langle j, k\rangle$ triangle with angles $j\alpha$, $k\alpha$, and $\pi - (j + k)$, for $\alpha = \arccos(r/2s)$, and all primitive integral $\langle j, k\rangle$ triangles are of this form for some such r and s.

Proof: By the proof of Proposition 1, for each r and s satisfying the conditions of the theorem, $\langle p_j(r/s),\ p_k(r/s),\ p_{j+k}(r/s)\rangle$ is a rational sided $\langle j, k\rangle$ triangle with the required angles. By Proposition 2, any similarity class of $\langle j, k\rangle$ triangles that includes a triangle with rational sides includes a triangle of this form for some r and s satisfying the hypotheses of the theorem. Our proposed triangle is clearly integer sided, and the definition of the P_n implies that it is similar to this one by a scale factor of s^{j+k-1}. Therefore, we need only prove that it is primitive. By Lemma 3(a), it suffices to show that, if u and v are positive integers with $(u, v) = 1$, then $(P_u(r, s), P_v(r, s)) = 1$. If $(u, v) = 1$, there are positive integers m and n such that mu and nv are consecutive integers. Then $(P_{mu}(r, s), P_{nv}(r, s)) = 1$ by Lemma 3(b). But, as noted above, $P_u|P_{mu}$ and $P_v|P_{nv}$. Thus, $(P_u(r, s), P_v(r, s)) = 1$, as required.

Example 1: To illustrate Theorem 1, we shall determine all primitive integral $\langle 3, 1\rangle$ triangles with no side longer than 100. Using Theorem 1, we know that they are of the form $\langle s(r^2 - s^2),\ s^3,\ r^3 - 2rs^2\rangle$ for r and s relatively prime positive integers with $\sqrt{2}/2 < r/2s < 1$. Since one side is s^3 and we are looking for those with sides no greater than 100, we must have $s = 1, 2, 3,$ or 4. For $s = 1$, we would need $\sqrt{2} < r < 2$, which is not possible. For $s = 2$, we need $2\sqrt{2} < r < 4$, which is only possible for $r = 3$ and which gives us the triangle $\langle 10, 8, 3\rangle$. For $s = 3$, we need $3\sqrt{2} < r < 6$, which is only possible for $r = 5$ and which gives us the triangle $\langle 48, 27, 35\rangle$. For $s = 4$, we need $4\sqrt{2} < r < 8$, which is only possible for $r = 6, 7$. But 6 is not relatively prime to 4 and $r = 7$ gives us the triangle $\langle 132, 64, 119\rangle$, two sides of which are too large.

References

1. G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. 4th ed. Oxford: Oxford University Press, 1960, pp. 190-91.
2. R. S. Luthar. "Integer-Sided Triangles with One Angle Twice Another." *The College Mathematics Journal 15.3* (1984):55-56.

*****