CONTINUED POWERS AND ROOTS

Dixon J. Jones
University of Alaska Fairbanks, Fairbanks, AK 99775-1110
(Submitted December 1988)

1. Introduction

For select real values of p and for real x_i, the expression

$$(1) \qquad \lim_{k\to\infty} x_0 + (x_1 + (x_2 + (\cdots + (x_k)^p \cdots)^p)^p)^p$$

is practically ubiquitous in mathematics. For instance, (1) represents nothing more than the old familiar $\Sigma_{k=0}^{\infty} x_k$ when $p = 1$. When $p = -1$, it becomes a novel notation for the continued fraction

$$x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots}}}$$

When $p = 0$, the expression is identically 1 (provided that the terms are not all 0).

Not quite ubiquitous, but certainly not rare, is the case $p = 1/2$, in which (1) becomes

$$(2) \qquad \lim_{k\to\infty} x_0 + \sqrt{x_1 + \sqrt{x_2 + \sqrt{\cdots + \sqrt{x_k}}}},$$

a form variously known as an "iterated radical," "infinite radical," "nested root," or "continued root." The literature reveals an assortment of problems involving (2) but only a smattering of other direct references. Of the few treatments of nested square roots as a research topic, one of the sharpest and most thorough is a paper by A. Herschfeld from 1935 [4], wherein he refers to (2) as a "right infinite radical" and derives necessary and sufficient conditions for its convergence. Recently, some of Herschfeld's results have been independently rediscovered [10].

A mathematical construct which includes infinite series, continued fractions, and infinite nested radicals as special cases ought to merit serious investigation. On the other hand, cases of (1) for other powers, for instance $p = 2$, seem likely to produce little more than irritating thickets of nested parentheses, and integer x_k clearly cause rapid divergence. [Herschfeld mentions the form (1), calls it a "generalized right infinite radical," notes the cases $p = 1$ and $p = -1$, states without proof what amounts to a necessary and sufficient condition for the convergence of (1) for $0 < p < 1$, and drops the subject there.] Yet, surprisingly, it turns out that (1) may converge even for very large p; even more surprisingly, there is a sense in which the convergence gets "better" the larger p grows.

In this article we gather and derive some basic properties of expression (1), especially its necessary and sufficient conditions for convergence. (For logistical reasons, we will deal only with positive powers p and nonnegative terms x_k; negative powers, complex terms, and interconnections between the variations represent unmapped territories which appear to be inhabited by interesting results.) We note the peculiar fickleness of infinite series in this context, and we conclude with a few comments interpreting (1) as a special composition of functions.

2. Definitions, Notation, and Qualitative Aspects

Given a sequence $\{x_n | n = 0, 1, 2, \ldots\}$ of real numbers (called *terms*), and given a real number p, define a sequence $\{y_n\}$ by

$$y_k = \mathop{C}_{i=0}^{k}(p, x_i) = x_0 + (x_1 + (x_2 + (\cdots + (x_k)^p \cdots)^p)^p)^p. \tag{3}$$

The limit of y_k as $k \to \infty$ will be called a *continued* (p^{th}) *power*, denoted by $C_{i=0}^{\infty}(p, x_i)$. If the limit exists, the continued power will be said to converge to that limit. (We do not insist that the limit be real, although it will be in what follows, given the assumption of positive terms and powers.) Borrowing from the jargon of continued fractions, $C_{i=0}^{k}(p, x_i)$ will be called the k^{th} *approximant* of the continued power. With the intent of both emphasizing and streamlining their retrograde associativity, we will make a slight deviation from standard notation and write continued powers and their k^{th} approximants, respectively, as

$$\mathop{C}_{i=0}^{\infty}(p, x_i) = x_0 + {}^{p}(x_1 + {}^{p}(x_2 + \cdots))$$

and

$$\mathop{C}_{i=0}^{k}(p, x_i) = x_0 + {}^{p}(x_1 + {}^{p}(\cdots + {}^{p}(x_k) \cdots)).$$

Implicit in this notation is the convention ${}^{p}(x) = x^p$, and the raising of quantities to powers will be effected both ways. For $j \geq 1$, we will call

$$\mathop{C}_{i=j}^{\infty}(p, x_i) = x_j + {}^{p}(x_{j+1} + {}^{p}(x_{j+2} + \cdots))$$

and

$$\mathop{C}_{i=j}^{k}(p, x_i) = x_j + {}^{p}(x_{j+1} + {}^{p}(\cdots + {}^{p}(x_k) \cdots))$$

the *truncation at* x_j of a continued power and of its k^{th} approximant, respectively. If the arguments p and x_i are understood in a given discussion, then $C_{i=j}^{k}(p, x_i)$ will be shortened to C_j^k. Note that

$$\mathop{C}_{k}^{k} = x_k \quad (k \geq 0),$$

$$\mathop{C}_{j}^{k} = x_j + {}^{p}\left(\mathop{C}_{j+1}^{k}\right) \quad (0 \leq j < k).$$

In the event that $p = 1/m$, m a positive integer [or, more loosely, for $m \in (1, \infty)$], we may use the notation developed in [10]:

$$\mathop{C}_{i=0}^{\infty}(p, x_i) = x_0 + \sqrt[m]{(x_1 + \sqrt[m]{(x_2 + \sqrt[m]{(\cdots}}}$$

and will call such an expression a *continued root* (dropping the m, of course, when $m = 2$).

The contrary associativity of a continued power is at the outset perhaps its most prominent and daunting feature. Not only must the evaluation of a finite approximant be performed from right to left, but the kth approximant cannot in general be obtained as a simple function of the $(k-1)^{\text{st}}$; that is, there is in general no simple recursion formula relating C_0^{k-1} to C_0^k. To manipulators of infinite series and continued fractions, this annoyance is less severe than it is to us, because the essentially linear and fractional nature of series and continued fractions permits the elimination of nested parentheses. For most continued powers, however, nonlinearity will subvert or preclude such simplification.

Since computation of the k^{th} approximant "begins" at x_k and "ends" at x_0, one might say that continued powers "end, but never begin" as the number of terms increases without bound. This is in stark contrast to most other infinite constructs (borne for the most part by truly iterated processes) which "begin, but never end." To have an end, but no beginning, seems rather bizarre; perhaps this is because our intuition, abstracted from the natural world, prefers infinite processes with finite origins. After all, anyone who is born can wish never to die, but what sense can be made of the possibility of dying, having never been born? For now, we will accept the informal idea of expressions that "end, but never begin" without dwelling on its deeper implications, lest by sheer grammatical duality the familiar processes that "begin, but never end" come to look equally doubtful.

3. Continued Powers of Constant Terms

Continued powers turn up in the literature often as continued square roots having constant terms, as in the formula (mentioned in [8]) for the golden ratio

$$\phi = \frac{1 + \sqrt{5}}{2} = \sqrt{}(1 + \sqrt{}(1 + \sqrt{}(1 + \sqrt{}(\cdots$$

Such expressions invite consideration of continued powers of the form

$$\overset{\infty}{\underset{i=0}{\mathrm{C}}}(p,\ a) = a + {}^{p}(a + {}^{p}(a + \cdots\ )).$$

For a given $p > 0$, what values of $a \geq 0$ (if any) will make this continued power converge?

To answer this question, we conjure up an insight so useful that in one way or another it makes possible all of our later results: namely, *the order of operations can be reversed in a continued power of constant terms*. That is, the evaluation of a finite approximant may be performed by associating either to the right or to the left when all the terms are equal, as the following construction demonstrates:

$$\begin{aligned} a &= a \\ a + {}^{p}(a) &= (a)^{p} + a \\ &\vdots \\ a + {}^{p}(\cdots + {}^{p}(a + {}^{p}(a))\ \cdots) &= (\cdots\ ((a)^{p} + a)^{p} + \cdots)^{p} + a \end{aligned} \tag{4}$$

where each side of the last line has the same number of terms. *Note that this does not work if the terms are not equal*. If you index the terms as you add them, you will find that neither the left- nor right-hand expressions are approximants of a continued power.

As mentioned in Section 2, associativity in the "wrong" direction is the main impediment to the study of continued powers in general. The appeal of the present situation lies in the fact that a continued power of constant terms is equivalent to a form whose associativity proceeds in the "right" direction, and whose convergence can be studied using known techniques. The tool we will make most use of is the algorithm known in numerical analysis as "successive approximation" or "fixed-point iteration"; for those whom it may benefit, we briefly synopsize this algorithm and its properties. In fixed-point iteration, a generating function g is defined on an interval I, a starting point w_0 is chosen in I, and a sequence $\{w_k\}$ is generated by $w_k = g(w_{k-1})$ for $k = 1, 2, 3, \ldots$. The sequence $\{w_k\}$ converges to an (*attracting*) *fixed point* λ in I [with the property that $g(\lambda) = \lambda$], provided that g and I satisfy certain conditions. For our purposes the following conditions due to Tricomi (mentioned in [3]) will suffice, although others are known (cf. [5]):

(i) $g(x)$ must be continuous on the (closed, half-open, or open) interval I;

(ii) there must exist a number $\lambda \in I$ such that $g(\lambda) = \lambda$;

(iii) $|g(x) - \lambda| < |x - \lambda|$ for all $x \in I$, $x \neq \lambda$.

Despite notational vagaries, it is no secret ([7], [9]) that, for $p = 1/2$, the expression $(((a)^p + a)^p + a)^p + \cdots$ is simply an "unabbreviated" fixed-point algorithm generated by $g(x) = g_a(x) = x^p + a$ at the starting point $x = 0$. Extending this interpretation to the general case, we invoke the identity (4) to claim that the convergence of $C_{i=0}^{\infty}(p, a)$ depends only on $g_a(x)$ and a suitable interval I containing the starting point $x = 0$ and the fixed point λ. In fact, C_0^{∞} converges just when g and I conform to conditions (i), (ii), and (iii) above. With this strategy in hand we obtain

Theorem 1: The continued p^{th} power with nonnegative constant terms $x_n = a$ converges if and only if

$$\begin{aligned} &a \geq 0 && \text{for } 0 < p < 1; \\ &a = 0 && \text{for } p = 1; \text{ and} \\ &0 \leq a \leq R && \text{for } p > 1 \end{aligned}$$

where

$$R = \sqrt[p-1]{\frac{(p-1)^{p-1}}{p^p}}$$

The set $[0, \infty)$ will be called the *interval of convergence* for a continued p^{th} power, $0 < p < 1$. Likewise $\{0\}$ and $[0, R]$ will be the intervals of convergence for $p = 1$ and $p > 1$, respectively.

Proof: The case $p = 1$ is trivial, since the only value of a for which $\sum_{i=0}^{\infty} a$ is finite is $a = 0$, and $R = 0$ when $p = 1$. Indeed, $C_{i=0}^{\infty}(p, a)$ converges whenever $a = 0$ for any $p > 0$.

For $g_a(x) = x^p + a$ and $p > 0$, continuity is not an issue for x and a in R^+. Condition (i) is satisfied by any positive interval.

Condition (iii) is fulfilled for $0 < p < 1$ and $p > 1$, since in both cases the function $g_a(x) = x^p + a$ is strictly increasing, and it is easily shown that either $\lambda > g_a(x) > x$ or $\lambda < g_a(x) < x$ for $x \neq \lambda$ in the interval(s) I which pertain.

The remainder of the proof, then, involves determining those intervals I and establishing the existence of $\lambda \in I$ for positive $p \neq 1$. Because the functions involved are very well-behaved, we offer remarks about their graphs rather than detailed derivations of their properties. Essentially, the problem is to determine how far a power function can be vertically translated so that it always possesses an attracting fixed point.

$0 < p < 1$. The curve $y = g_a(x) = x^p + a$ (typified by $y = \sqrt{x} + a$) is strictly increasing, concave downward, and vertically translated $+a$ units. For $a > 0$, take $I = [0, \infty)$. From a graph, it is clear that $y = g_a(x)$ intersects $y = x$ exactly once in I, at the point $x = \lambda = g_a(x)$. (For a treatment of this case when $p = 1/2$ and a is complex, see [11].)

$p > 1$. Here the curve $y = g_a(x)$ is exemplified by $y = x^2 + a$; it is concave upward, strictly increasing, and elevated a units. There is a point $a = R$ at which $y = g_a(x)$ is tangent to $y = x$; for $a > R$, the two curves do not intersect; hence, no $\lambda = g_a(\lambda)$ exist.

When $a = R$, λ is the point of tangency of $y = g_R(x)$ and $y = x$. The derivative of $g_R(x)$ is 1 at $x = \lambda$, whereby $\lambda = X = (1/p)^{1/(p-1)}$. Then, from $\lambda = g_R(\lambda) = \lambda^p + R$ and $\lambda = X$, we find

$$R = X - X^p = \left(\frac{1}{p}\right)^{\frac{1}{p-1}} - \left(\frac{1}{p}\right)^{\frac{p}{p-1}} = \left(\frac{1}{p}\right)^{\frac{p}{p-1}}(p-1) = \sqrt[p-1]{\frac{(p-1)^{p-1}}{p^p}}.$$

This form for R was chosen to foreshadow a recurrent theme in the field of continued powers, namely the persistent appearance of expressions of the form A^A/B^B. At any rate, for $a = R$, take $I = [0, X]$.

Finally, when $0 \le a < R$, $y = g_a(x)$ intersects $y = x$ at two points lying on either side of the point X. Take $I = [0, X]$, so that the single intersection point less than X is the point $\lambda \in I$ satisfying condition (ii). We have shown that $g_a(x)$ generates convergent fixed-point algorithms over $I = [0, X]$ for $0 \le a \le R$, which ends the proof.

Theorem 1 reveals that, for instance

$$\underset{i=0}{\overset{\infty}{C}}(2, a) = a + {}^2(a + {}^2(a + \cdots))$$

converges for any $a \in [0, 1/4]$; the proof shows that

$$\underset{i=0}{\overset{\infty}{C}}\left(2, \frac{1}{4}\right) = \frac{1}{2}.$$

One may show that as $p \to \infty$ the point $R \to 1$, hence the interval of convergence grows larger as p increases beyond 1. In this context, we can reasonably say that the convergence of a continued p^{th} power gets "better" as p grows large, and is "worst" for the famous case $p = 1$, namely infinite series.

4. Continued Powers of Arbitrary Terms; $0 < p < 1$

The first discussion of the convergence of the continued square root

$$\underset{i=0}{\overset{\infty}{C}}\left(\frac{1}{2}, x_i\right) = x_0 + \sqrt{}(x_1 + \sqrt{}(x_2 + \sqrt{}(\cdots$$

appears to have been made in 1916 by Pólya & Szegö [8], who showed that it converges or diverges accordingly as

$$\limsup_{n \to \infty} \frac{\log \log x_n}{n}$$

is less than or greater than log 2. This result was encompassed by a theorem of Herschfeld, which gives a necessary and sufficient condition for the convergence of a continued square root and which easily generalizes to the main theorem of this section.

Theorem 2: For $0 < p < 1$, the continued p^{th} power with terms $x_n \ge 0$ converges if and only if $\{x_n^{p^n}\}$ is bounded.

This follows simply by substitution of $1/p^{\text{th}}$ roots for square roots in Herschfeld's proof of the case $p = 1/2$. In lieu of a proof by plagiarism, we merely convey the proof's salient features; and to that end, let us take a moment to establish three useful properties of approximants and their truncations. (Remember that $\{x_n\}$ is nonnegative and p is positive in what follows.) First, successive truncations of the approximant C_0^k conform to the inequality

$$\underset{j}{\overset{k}{C}} \ge \left(\underset{j+1}{\overset{k}{C}}\right)^p \quad (0 \le j \le k - 1)$$

which follows from $C_j^k = x_j + (C_{j+1}^k)^p$. Furthermore, the approximants form a nondecreasing sequence:

$$\underset{0}{\overset{k+1}{C}} \ge \underset{0}{\overset{k}{C}} \quad (k \ge 0).$$

To see this, start with $x_k + {}^p(x_{k+1}) \ge x_k$ and construct each approximant backwards to x_0. Finally, from the formula

$$\overset{k}{\underset{0}{C}} = x_0 + {}^p\!\left(x_1 + {}^p\!\left(\cdots + {}^p\!\left(x_{j-1} + {}^p\!\left(\overset{k}{\underset{j}{C}}\right)\right) \cdots \right)\right)$$

it is clear that a continued power converges if any truncation converges.

The necessity of Theorem 2 is easily proved by applying the inequality for successive truncations n times to C_0^n and letting $n \to \infty$:

$$\overset{n}{\underset{0}{C}} \geq \left(\overset{n}{\underset{n}{C}}\right)^{p^n} = x_n^{p^n}$$

$$\overset{\infty}{\underset{0}{C}} \geq \lim_{n\to\infty} x_n^{p^n}.$$

C_0^∞ converges; hence, $\{x_n^{p^n}\}$ is bounded.

On the other hand, suppose there is an $M > 0$ such that $x_n^{p^n} \leq M$ for all $n \geq 0$ or, equivalently, $x_n \leq M^{p^{-n}}$. With this, one can construct the inequality

$$x_0 + {}^p(x_1 + {}^p(\cdots + {}^p(x_n) \cdots)) \leq M + {}^p(M^{p^{-1}} + {}^p(\cdots + {}^p(M^{p^{-n}}) \cdots)).$$

Multiplying the right side by M/M and distributing the denominator through the successive parentheses results in

$$\overset{n}{\underset{i=0}{C}}(p,\ x_i) \leq M[1 + {}^p(1 + {}^p(\cdots + {}^p(1) \cdots))]$$

or

$$\overset{n}{\underset{i=0}{C}}(p,\ x_i) \leq M \overset{n}{\underset{i=0}{C}}(p,\ 1).$$

The continued root on the right converges as $n \to \infty$, because 1 is in the set of constants for all continued roots. The nondecreasing approximants on the left are therefore bounded; hence, $C_{i=0}^\infty(p,\ x_i)$ converges, which finishes the proof.

The condition of Theorem 3 is met by most common sequences. An example of a divergent continued root is

$$\overset{\infty}{\underset{i=0}{C}}\left(\frac{1}{3},\ 2^{4^i}\right) = 2 + \sqrt[3]{(2^4 + \sqrt[3]{(2^{16} + \sqrt[3]{(2^{64} + \sqrt[3]{(\ \cdots}}}}$$

where the sequence of terms fails the "upper bound" test: $(2^{4^n})^{p^n} = 2^{(4/3)^n} \to \infty$.

5. Continued Powers of Arbitrary Terms; $p \geq 1$

As p exceeds the critical value 1, continued p^{th} powers converge with markedly lower enthusiasm. They behave stubbornly, although not pathologically—for, given the hypotheses of this discourse, we are favored at least with a nondecreasing sequence of approximants—and in one sense the most reticent examples are infinite series ($p = 1$). In this section we will show that, among other things, the better-known convergence tests for series are just limiting cases of conditions which hold for general continued p^{th} powers ($p > 1$).

For instance, it is common knowledge that, if an infinite series converges, then its n^{th} term must approach zero. The analogous property for continued powers is summarized in

Theorem 3: For $p > 1$, the continued p^{th} power with terms $x_n \geq 0$ and interval of convergence $[0,\ R]$ converges if $\limsup x_n < R$. For $p \geq 1$, it diverges if $\liminf x_n > R$.

Proof: We first prove the latter assertion. If $\liminf x_n = B > R$, then for each $\varepsilon > 0$ there is a natural number N such that $B - \varepsilon < x_n$ for all $n \geq N$. In particular, choose $\varepsilon = \varepsilon_0 > 0$ such that $R < B - \varepsilon_0 < x_n$, and for convenience, set $v = B - \varepsilon_0$. Then use $v < x_n$ for all $n \geq N$ to construct

$$v + {}^p(v + {}^p(\cdots + {}^p(v) \cdots)) < x_N + {}^p(x_{N+1} + {}^p(\cdots + {}^p(x_n) \cdots)).$$

More compactly we have, in the limiting case,

$$\overset{\infty}{\underset{i=N}{C}}(p,\ v) \le \overset{\infty}{\underset{i=N}{C}}(p,\ x_i).$$

But $C_{i=N}^{\infty}(p,\ v)$ diverges, because $v = B - \varepsilon_0$ is greater than R and not in the interval of convergence. Therefore, the truncation $C_{i=N}^{\infty}(p,\ x_i)$ diverges, and likewise the entire continued power.

A similar argument shows that, if $\limsup x_n = B < R$, the continued power converges. However, if $R = 0$, we would be assuming that $\limsup x_n = B < 0$, which for a nonnegative sequence is a malfeasance. By excluding the case $p = 1$ (for which $R = 0$), we salvage this argument and complete the proof.

We come now to a situation wherein continued powers show substantially greater resistance to examination. The deep questions of our present line of inquiry involve powers greater than one and terms x_n for which

$$\liminf x_n \le R \le \limsup x_n.$$

One of the simplest examples with these features is the continued square

$$\overset{\infty}{\underset{i=0}{C}}(2,\ t_i),$$

where we have nonnegative constants a and b such that $t_{2i+1} = a$, $t_{2i} = b$, and $a \le 1/4 \le b$ ($R = 1/4$ for a continued square). That is,

$$\overset{\infty}{\underset{i=0}{C}}(2,\ t\ ) = b + {}^2(a + {}^2(b + {}^2(a + \cdots))).$$

Our approach to this example parallels the development of Section 3. The problem of "backwards" associativity is overcome by the identities

$$b + {}^2(a + {}^2(\cdots + {}^2(a + {}^2(b))\ \cdots)) = ((\cdots((b)^2 + a)^2 + \cdots)^2 + a)^2 + b, \tag{5}$$

where each side has the same odd number of terms, and

$$b + {}^2(a + {}^2(\cdots + {}^2(b + {}^2(a))\ \cdots)) = ((\cdots((a)^2 + b)^2 + \cdots)^2 + a)^2 + b, \tag{6}$$

where each side has the same even number of terms. The right-hand sides of these equations can each be thought of as an unabbreviated fixed-point algorithm generated by the function $g_{a,b}(x) = (x^2 + a)^2 + b$; in equation (5) the starting point is $x = b$, while in (6) it is $x = 0$. We want this algorithm to converge to the same limit regardless of the point at which it starts. Under our hypotheses, $g_{a,b}$ is positive, strictly increasing, and "concave upwards" in $\mathbf{R}^+$; a and b are not both 0; thus, it follows that there is a unique point in $\mathbf{R}^+$ where the derivative of $g_{a,b}$ equals 1. This leads to the equation $4x^3 + 4ax - 1 = 0$, having a single positive real solution which we call γ (stated explicitly below).

The convergence of the fixed-point algorithm using $g_{a,b}$ can now be assured. For $b = \gamma - (\gamma^2 + a)^2$, the unique attracting fixed point in $\mathbf{R}^+$ of $g_{a,b}$ is the point of tangency of $y = g_{a,b}(x)$ and $y = x$. When $b < \gamma - (\gamma^2 + a)^2$, $y = g_{a,b}(x)$ intersects $y = x$ in two points lying on either side of $x = \gamma$, and the left one is the desired attracting fixed point. The interval $I = [0,\ \gamma]$ maps into itself, and since both 0 and b are contained in I, they may be used as starting points for a convergent fixed-point algorithm using $g_{a,b}$. Thus, we are led to the following

Proposition: For $0 \le a \le 1/4 \le b$, the continued square

$$b + {}^2(a + {}^2(b + {}^2(a + \cdots)))$$

converges if and only if $b \le \gamma - (\gamma^2 + a)^2$, where

$$\gamma = \sqrt[3]{\frac{1}{8} + \sqrt{\frac{1}{64} + \frac{a^3}{27}}} + \sqrt[3]{\frac{1}{8} - \sqrt{\frac{1}{64} + \frac{a^3}{27}}}.$$

(The reader may find it entertaining to show by this Proposition that $b = 1/4$ if $a = 1/4$, as Theorem 1 requires.) This is not a particularly graceful conclusion to an admittedly rough sketch. But not much more elegant, and considerably less specific, is the generalization to powers other than 2, via the same argument.

Theorem 4: Given $p > 1$, interval of convergence $[0, R]$, and $0 \le a \le R \le b$, the continued p^{th} power

$$b + {}^p(a + {}^p(b + {}^p(a + \cdots)))$$

converges if and only if $b \le \gamma - (\gamma^p + a)^p$, where γ is the unique root in $\mathbf{R}^+$ of $p^2(x^{p+1} + ax)^{p-1} - 1 = 0$.

And so the simplest continued power for which $\liminf x_n \le R \le \limsup x_n$ leads to a result whose application will in most cases require solution of an equation by numerical approximation. Worse yet, note that Theorem 4 has virtually no relevance to

$$b + {}^p(b + {}^p(a + {}^p(b + {}^p(b + {}^p(a + \cdots)))))$$

or to similar constructions in which various arrangements of two constants make up the sequence of terms. Such apparitions are manageable to the extent that we *can* find generating functions for equivalent fixed-point algorithms; these functions and their derivatives, however, are not likely to be pleasant to work with, especially for noninteger p.

On the other hand, one should not be left believing that the situation is near hopeless when $\liminf x_n \le R \le \limsup x_n$. For instance, satisfying results are attainable for a continued power whose terms monotonically decrease to R. Subsumed by this special case are (not necessarily convergent) infinite series whose terms decrease to 0. Just as the ratio of consecutive terms sometimes imparts useful information about the convergence of series, so too does a kind of "souped-up" ratio test apply to continued p^{th} powers. In fact, the continued powers test almost reduces to d'Alembert's ratio test for series as $p \to 1$, but the precarious nature of infinite sums considered as special continued powers causes an interesting and instructive discrepancy.

Theorem 5: For $p > 1$, the continued p^{th} power with terms $x_n > 0$ converges if

$$\frac{(x_{n+1})^p}{x_n} \le \frac{(p-1)^{p-1}}{p^p}$$

for all sufficiently large values of n.

Proof: Assume the validity of the ratio test (for $n \ge 0$, without loss of generality) in the form $(x_{n+1})^p \le cx_n$, where $c = (p-1)^{p-1}/p^p$. Using this inequality, a proof by induction on the index k ($k \le n$) shows that

$$\underset{n-k}{\overset{n}{\mathbf{C}}} \le (x_{n-k})[1 + c\ {}^p(1 + c\ {}^p(\cdots + c\ {}^p(1 + c)\ \cdots))], \tag{7}$$

where the number of c's on the right is k. When $k = n$, (7) becomes

$$\underset{0}{\overset{n}{\mathbf{C}}} \le x_0[1 + c\ {}^p(1 + c\ {}^p(\ \cdots + c\ {}^p(1 + c)\ \cdots))], \tag{8}$$

where the number of c's is now n. The right side of (8) contains a variation on a continued power of constants, equivalent to an unabbreviated fixed-point algorithm generated by the function $g(x) = 1 + cx^p$ at the starting point $x = 0$:

$$(9) \qquad 1 + c^{\,p}(1 + c^{\,p}(\cdots + c^{\,p}(1 + c)\ \cdots))$$
$$= ((\cdots\ (c + 1)^p\ c + \cdots)^p\ c + 1)^p\ c + 1,$$

where both sides are of equal length. By applying the conditions (i), (ii), and (iii) from Section 3, this algorithm can be shown to converge on the interval $I = [0,\ p/(p - 1)]$, which just manages to include both the starting point $x = 0$ and the fixed point $\lambda = p/(p - 1)$. Thus, the right side of (9) converges in the limiting case to $p/(p - 1)$, which when combined with (8) shows that

$$(10) \qquad \lim_{n\to\infty} \overset{n}{\underset{0}{C}} \le x_0\left(\frac{p}{p - 1}\right).$$

We therefore infer the congruence of C_0^{∞}, which completes the proof.

The continued square $C_{i=0}^{\infty}(2,\ 4^{(2^{-i}-1)})$ is an example of a continued power which converges by the test of Theorem 5. The sequence of terms

$$\{1,\ 4^{-1/2},\ 4^{-3/4},\ 4^{-7/8},\ \ldots\}$$

satisfies the inequality $(x_{n+1})^2/x_n \le 1/4$; in fact, equality holds for all n. That the ratio test is not necessary for convergence, even when the terms decrease monotonically, is demonstrated by

$$\overset{\infty}{\underset{i=0}{C}}\left(2,\ \frac{1}{2} + 2^{-i}\right),$$

which converges by comparison with the other continued square mentioned above. (The proof depends on the inequality

$$\frac{1}{4} + 2^{-(n+2)} < 4^{(2^{-n}-1)},$$

whose verification is a mildly interesting exercise in its own right.) The terms $x_n = 1/4 + 2^{-n}$ satisfy the necessary condition $\liminf x_n = 1/4$, but fail the ratio test for all n because

$$(x_{n+1})^2/x_n = \frac{1}{4} + 1/(2^{2n} + 2^{n+2}).$$

Since $(p - 1)^{p-1}/p^p \to 1$ as $p \to 1$, Theorem 5 seems to tell us that an infinite series converges if $x_{n+1}/x_n \le 1$. The many erroneous aspects of this conclusion arise because the fixed point of $g(x) = 1 + cx^p$, namely $\lambda = p/(p - 1)$, ceases to be finite when $p = 1$. Thus, in the inequality (10), the series is not bounded, and the construction used to prove the ratio test becomes indeterminate.

6. Continued Powers as Function Compositions

The analytic theory of continued fractions has long recognized that continued fractions, infinite series, and even infinite products can be defined in the complex plane by means of the composition

$$(11) \qquad F_k(w_0) = f_0 \circ f_1 \circ \cdots \circ f_k(w_0)$$

of linear fractional transformations

$$f_k(w) = \frac{a_k + c_k w}{b_k + d_k w}, \quad k = 0,\ 1,\ 2,\ \ldots,$$

by suitable choices of a_k, b_k, c_k, and d_k [6]. Many other constructs can be defined similarly using different functions for the f_k. For instance $f_k(w) = a_k + tw$ and $w_0 = 0$ produces polynomials in t. For real x, $f_k(x) = (a_k)^x$, with $a_k > 0$, $k = 0, 1, 2, \ldots$, generates what is sometimes called a "tower" or a "continued exponential":

$$F_k(1) = a_0^{a_1^{a_2^{\cdot^{\cdot^{\cdot^{a_k}}}}}}$$

where evaluation is made from the top down ([1], [2]).

This paper has investigated the limiting behavior of (11) when

$$f_k(x) = x_k + x^p, \text{ with } x \geq 0,\ p > 0, \text{ and } x_k \geq 0 \text{ for } k = 0, 1, 2, \ldots .$$

The order of composition in (11) is synonymous with the problematical associativity of continued powers. In retrospect, our progress depended on establishing the convergence of (11) for the special case $f_0 = f_1 = \cdots = f_k = g$, where we variously used $g(x) = x^p + a$, $g(x) = (x^p + a)^p + b$, and $g(x) = 1 + cx^p$. In these cases the composition (11) reduces to

$$F_k(0) = g \circ g \circ \cdots \circ g(0)$$

whose handy recursion formula

$$F_k(0) = g \circ F_{k-1}(0)$$

paves the way for conquest by fixed-point algorithms. This method promises to be helpful in exploring continued negative powers and other function-compositional objects that distinguish themselves by uncooperatively nesting their operations.

Acknowledgments

The author is indebted to M. Getz, G. Gislason, and J. P. Lambert of the University of Alaska Fairbanks for their comments and advice; to D. R. DeWitt for early experiences with continued roots; and to P. R. Halmos for benign and compact encouragement.

References

1. D. F. Barrow. "Infinite Exponentials." *Amer. Math. Monthly* *43* (1936):150-160.
2. M. Creutz & R. M. Sternheimer. "On the Convergence of Iterated Exponentiation—I." *Fibonacci Quarterly* *18* (1980):341-47.
3. J. B. Diaz & F. T. Metcalf. "On the Set of Subsequential Limit Points of Successive Approximations." *Trans. Amer. Math. Soc.* *135* (1969):459-85.
4. A. Herschfeld. "On Infinite Radicals." *Amer. Math. Monthly* *42* (1935):419-429.
5. E. Isaacson & H. B. Keller. *Analysis of Numerical Methods*. New York: John Wiley, 1966, pp. 85-91.
6. W. B. Jones & W. J. Thron. *Continued Fractions: Analytic Theory and Applications*. London: Addison-Wesley, 1980, p. 11.
7. C. S. Ogilvy. "To What Limits Do Complex Iterated Radicals Converge?" *Amer. Math. Monthly* *77* (1980):388-89.
8. G. Pólya & G. Szegö. *Problems and Theorems in Analysis*. Vol. I. New York: Springer-Verlag, 1972.
9. E. J. Purcell & D. Varberg. *Calculus with Analytic Geometry*. 4th ed. Englewood Cliffs, N.J.: Prentice Hall, 1984, p. 448.
10. W. S. Sizer. "Continued Roots." *Math. Magazine* *59* (1986):23-27.
11. P. L. Walker. "Iterated Complex Radicals." *Math. Gazette* *67* (1983):269-73.

*****