SUBSETS WITHOUT UNIT SEPARATION AND PRODUCTS OF FIBONACCI NUMBERS

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1. Introduction

It is well known that the Fibonacci numbers are intimately related to subsets of $\{1, 2, 3, \ldots, n\}$ not containing a pair of consecutive integers. More precisely, let F_n denote the n^{th} Fibonacci number determined by the recurrence relation

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n \quad (n \geq 1).$$

Then the total number of subsets of $\{1, 2, 3, \ldots, n\}$ not containing a pair of consecutive integers is F_{n+2}. This result can also be expressed in terms of a well-known combinatorial identity. Kaplansky [2] showed that the number of k-subsets of $\{1, 2, 3, \ldots, n\}$ not containing a pair of consecutive integers is

$$\binom{n+1-k}{k}.$$

Consequently, summing over k we obtain the identity

$$\text{(1)} \qquad \sum_{k \geq 0} \binom{n+1-k}{k} = F_{n+2}.$$

In this paper we will derive a combinatorial identity expressing the square of a Fibonacci number and the product of two consecutive Fibonacci numbers in terms of the number of subsets of $\{1, 2, 3, \ldots, n\}$ without unit separation. Two objects are called *uniseparate* if they contain exactly one object between them. For example, the following pairs of integers are uniseparate: (1, 3), (2, 4), (3, 5), etc. Konvalina [3] showed that the number of k-subsets of $\{1, 2, 3, \ldots n\}$ not containing a pair of uniseparate integers is

$$\text{(2)} \qquad \begin{cases} \displaystyle\sum_{i=0}^{[k/2]} \binom{n+1-k-2i}{k-2i} & \text{if } n \geq 2(k-1), \\ 0 & \text{if } n < 2(k-1). \end{cases}$$

Let T_n denote the total number of subsets of $\{1, 2, 3, \ldots, n\}$ without unit separation. Then, summing over k, we have

$$\text{(3)} \qquad T_n = \sum_{k \geq 0} \sum_{i=0}^{[k/2]} \binom{n+1-k-2i}{k-2i}.$$

We will prove that if n is even then T_n is the square of a Fibonacci number; while, if n is odd T_n is the product of two consecutive Fibonacci numbers.

2. Main Result

Theorem: If $n \geq 1$, then
$$T_{2n} = F_{n+2}^2$$
$$T_{2n+1} = F_{n+2}F_{n+3}.$$

Proof: The following identities on summing every fourth Fibonacci number are needed in obtaining the result:

$$\sum_{j=1}^{n} F_{4j} = F_{2n+1}^2 - 1, \tag{4}$$

$$\sum_{j=1}^{n} F_{4j-2} = F_{2n}^2; \tag{5}$$

$$\sum_{j=1}^{n} F_{4j-3} = F_{2n-1}F_{2n}; \tag{6}$$

$$\sum_{j=1}^{n} F_{4j-1} = F_{2n}F_{2n+1}. \tag{7}$$

These identities are easily proved by induction and the following well-known Fibonacci identities (see Hoggatt [1]):

$$F_{n+1}^2 - F_{n-1}^2 = F_{2n},$$
$$F_nF_{n+1} - F_{n-1}F_{n-2} = F_{2n-1}.$$

Now, evaluating T_n in (3), we obtain

$$T_n = \sum_{k\geq 0} \sum_{i=0}^{[k/2]} \binom{n+1-k-2i}{k-2i} = \sum_{k=0}^{[(n+2)/2]} \sum_{i\geq 0} \binom{n+1-k-2i}{k-2i}.$$

Now, replacing k by $k + 2i$, since $k - 2i$ contributes zero to the sum, we obtain the key identity

$$T_n = \sum_{i\geq 0} \sum_{k=0}^{m} \binom{n+1-k-4i}{k}, \tag{8}$$

where $m = [(n+2)/2] - 2i$.

Next, we will apply (1) and the Fibonacci identities (4), (5), (6), and (7) to evaluate (8). First, identity (1) can be expressed as follows:

$$\sum_{k\geq 0} \binom{n+1-k}{k} = \sum_{k=0}^{[(n+1)/2]} \binom{n+1-k}{k} = F_{n+2}. \tag{9}$$

Replacing n by $n - 4i$, identity (9) becomes

$$\sum_{k=0}^{p} \binom{n+1-k-4i}{k} = F_{n+2-4i}, \tag{10}$$

where $p = [(n+1)/2] - 2i$.

To complete the proof, we will evaluate (8) based on whether $n \equiv 0, 1, 2,$ or 3 (mod 4).

Odd Case: If n is odd, then $[(n+2)/2] = [(n+1)/2]$, so $m = p$ and, applying (10) to (8), we have a sum involving every fourth Fibonacci number.

$$T_n = \sum_{i\geq 0} F_{n+2-4i}. \tag{11}$$

Case 1. $n \equiv 1 \pmod 4$

In this case we have $n + 2 = 4t - 1$ for some integer t. Substitute $t = (n+3)/4$ for n in (7) and apply to (11) to obtain

$$T_n = F_{(n+3)/2}F_{(n+3)/2+1}.$$

Since n is odd, replace n by $2n + 1$, and the desired result

$$T_{2n+1} = F_{n+2}F_{n+3}$$

is obtained.

Case 2. $n \equiv 3 \pmod 4$

In this case we have $n + 2 = 4t - 3$ for some t. Substitute $t = (n + 5)/4$ for n in (6) and apply to (11) to obtain

$$T = F_{(n+5)/2-1}F_{(n+5)/2}.$$

Replace n by $2n + 1$ and the result is the same as in the previous case:

$$T_{2n+1} = F_{n+2}F_{n+3}.$$

Even Case: If n is even, $m = p + 1$, and applying (10) to (8) we have

$$(12) \qquad T_n = \sum_{i\geq 0}\sum_{k=0}^{p+1}\binom{n+1-k-4i}{k}$$

$$= \sum_{i\geq 0}\sum_{k=0}^{p}\binom{n+1-k-4i}{k} + \sum_{i\geq 0}\binom{n+1-(p+1)-4i}{p+1}$$

$$= \sum_{i\geq 0}F_{n+2-4i} + \sum_{i\geq 0}\binom{n/2-2i}{n/2-2i+1}.$$

Observe that the last summation is zero except when $n/2 - 2i + 1 = 0$. That is, when $i = (n + 2)/4$ or $n + 2 \equiv 0 \pmod 4$. In this case, the last sum is 1.

Case 3. $n \equiv 2 \pmod 4$

Here $n + 2 = 4t$ for some t. Substitute $t = (n + 2)/4$ for n in (4) and apply to (12) to obtain

$$T_n = (F^2_{(n+4)/2} - 1) + 1 = F^2_{(n+4)/2}.$$

Since n is even, replace n by $2n$ and the desired result is obtained:

$$T_{2n} = F^2_{n+2}.$$

Case 4. $n \equiv 0 \pmod 4$

Here $n + 2 = 4t - 2$ for some t. Substitute $t = (n + 4)/4$ for n in (5) and apply to (12) to obtain

$$T_n = F^2_{(n+4)/2}.$$

Replace n by $2n$ and the result is the same as in the previous case.

Table 1

n	F_n	F_n^2	F_nF_{n+1}	T_n	n	F_n	F_n^2	F_nF_{n+1}	T_n
1	1	1	1	1	7	13	169	273	40
2	1	1	2	4	8	21	441	714	64
3	2	4	6	6	9	34	1156	1870	104
4	3	9	15	9	10	55	3025	4895	169
5	5	25	40	15	11	89	7921	12816	273
6	8	64	104	25	12	144	20736	33552	441

References

1. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.
2. I. Kaplansky. "Solution of the "Probleme des menages." *Bull. Amer. Math. Soc.* *49* (1943):784-85.
3. J. Konvalina. "On the Number of Combinations without Unit Separation." *J. Combin. Theory*, Ser. *A31* (1981):101-07.

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