ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

IMPORTANT NOTICE: There is a new editor of this department and a new address for all submissions.

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to DR. STANLEY RABINOWITZ; 12 VINE BROOK RD.; WESTFORD, MA 01886-4212 USA.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n, satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-694 *Proposed by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA*

Prove that $L_{2^n} \equiv 7 \pmod{40}$ for $n \geq 2$.

B-695 *Proposed by Russell Euler, Northwest Missouri State U., Maryville, MO*

Define the sequences $\{P_n\}$ and $\{Q_n\}$ by

$$P_0 = 0, \; P_1 = 1, \; P_{n+2} = 2P_{n+1} + P_n \quad \text{for } n \geq 0$$

and

$$Q_0 = 1, \; Q_1 = 1, \; Q_{n+2} = 2Q_{n+1} + Q_n \quad \text{for } n \geq 0.$$

Find a simple formula expressing Q_n in terms of P_n.

B-696 *Proposed by Herta T. Freitag, Roanoke, VA*

Let (a, b, c) be a Pythagorean triple with the hypotenuse $c = 5F_{2n+3}$ and $a = L_{2n+3} + 4(-1)^{n+1}$.

(a) Determine b.
(b) For what values of n, if any, is the triple primitive? [The elements of a primitive triple have no common factor.]

B-697 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Find a closed form for the sum

$$S_n = \sum_{k=1}^{n} \frac{q^{k-1}}{w_k w_{k+1}}$$

where $w_n \neq 0$ for all n and $w_n = pw_{n-1} - qw_{n-2}$ for $n \geq 2$, with p and q nonzero constants.

B-698 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Consider the sequence of real numbers $a_1, a_2, \ldots,$ where $a_1 > 2$ and

$$a_{n+1} = a_n^2 - 2 \quad \text{for } n \geq 1.$$

Find $\lim_{n \to \infty} b_n$, where

$$b_n = \frac{a_{n+1}}{a_1 a_2 \ldots a_n} \quad \text{for } n \geq 1.$$

B-699 *Proposed by Larry Blaine, Plymouth State College, Plymouth, NH*

Let a be an integer greater than 1. Define a function $p(n)$ by

$$p(1) = a - 1 \quad \text{and} \quad p(n) = a^n - 1 - \sum p(d) \quad \text{for } n \geq 2,$$

where $\sum$ denotes the sum over all d with $1 \leq d < n$ and $d|n$.

Prove or disprove that $n|p(n)$ for all positive integers n.

SOLUTIONS
edited by A. P. Hillman

Application of Generating Functions

B-670 *Proposed by Russell Euler, Northwest Missouri State U., Maryville, MO*

Evaluate $\sum_{n=1}^{\infty} \frac{nF_n}{2^n}$.

Solution by Russell Jay Hendel, Dowling College, Oakdale, NY

The generating function

$$F(x) \equiv \sum F_n x^n = -\frac{x}{x^2 + x - 1}$$

has radius of convergence α^{-1}. Differentiating both sides with respect to x and then multiplying by x gives:

$$\sum_{n=1}^{\infty} nF_n x^n = F(x) + (F(x))^2(2x + 1), \quad \text{for } |x| < \alpha^{-1}.$$

Therefore, letting $x = .5$ in the last equation, we find

$$\sum_{n=1}^{\infty} \frac{nF_n}{2^n} = 10.$$

Also solved by Richard André-Jeannin, Barry Booton, Paul S. Bruckman, Joe Howard, Hans Kappus, Joseph J. Kostal, Y. H. Harris Kwong, Alex Necochea, Bob Prielipp, Don Redmond, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Even Perfect Numbers Are Hexagonal and Triangular

B-671 *Proposed by Herta T. Freitag, Roanoke, VA*

Show that all even perfect numbers are hexagonal and hence are all triangular. [A perfect number is a positive integer which is the sum of its proper positive integral divisors. The hexagonal numbers are {1, 6, 15, 28, 45, ...} and the triangular numbers are {1, 3, 6, 10, 15, ...}.]

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

The formulas for the k^{th} triangular number T_k and the k^{th} hexagonal number H_k are

$$T_k = \frac{k(k+1)}{2} \quad \text{and} \quad H_k = k(2k-1) = T_{2k-1},$$

respectively. It is well known that every even perfect number n is of the form

$$n = 2^{p-1}(2^p - 1),$$

where $2^p - 1$ is prime; so n is the $(2^{p-1})^{\text{th}}$ hexagonal number, which is also triangular.

Also solved by Richard André-Jeannin, Charles Ashbacher, Paul S. Bruckman, Russell Euler, Russell Jay Hendel, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Proposal in 10•199, Solution in 11•181

B-672 *Proposed by Philip L. Mana, Albuquerque, NM*

Let S consist of all positive integers n such that $n = 10p$ and $n + 1 = 11q$, with p and q primes. What is the largest positive integer d such that every n in S is a term in an arithmetic progression a, $a + d$, $a + 2d$, ...?

Solution by Richard André-Jeannin, Sfax, Tunisia

Let n be a member of S. It is clear that $11(q-1) = 10(p-1)$; hence,

$$p = 11r + 1 \quad \text{and} \quad q = 10r + 1.$$

Since p, q are prime numbers, it is easily proved that r is even and $r \equiv 0 \pmod 3$. Hence, $r = 6s$, $p = 66s + 1$, $q = 60s + 1$, and the members of S are terms in the arithmetical progression $u_s = 660s + 10$.

Now we have

$$u_{10} = 10 \cdot 661,\ u_{10} + 1 = 11 \cdot 601,$$

and

$$u_{11} = 10 \cdot 727,\ u_{11} + 1 = 11 \cdot 661;$$

hence, u_{10} and u_{11} are members of S, and the largest d such that every n in S is in an arithmetical progression is $d = 660$.

Also solved by Charles Ashbacher, Paul S. Bruckman, Y. H. Harris Kwong, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Fibonacci Infinite Product

B-673 *Proposed by Paul S. Bruckman, Edmonds, WA*

Evaluate the infinite product $\prod_{n=2}^{\infty} \frac{F_{2n} + 1}{F_{2n} - 1}$.

Solution by Joseph J. Kostal, U. of Illinois at Chicago, IL

$$\prod_{n=2}^{\infty} \frac{F_{2n}+1}{F_{2n}-1} = \prod_{n=1}^{\infty} \left(\frac{F_{4n}+1}{F_{4n}-1} \cdot \frac{F_{4n+2}+1}{F_{4n+2}-1}\right)$$

$$= \prod_{n=1}^{\infty} \left(\frac{F_{2n-1}L_{2n+1}}{F_{2n+1}L_{2n-1}} \cdot \frac{F_{2n+2}L_{2n}}{F_{2n}L_{2n+2}}\right)$$

$$= \prod_{n=1}^{\infty} \left(\frac{F_{2n-1}F_{2n+2}}{F_{2n}F_{2n+1}} \cdot \frac{L_{2n}L_{2n+1}}{L_{2n-1}L_{2n+2}}\right)$$

$$= \prod_{n=1}^{\infty} \frac{F_{2n-1}F_{2n+2}}{F_{2n}F_{2n+1}} \cdot \prod_{n=1}^{\infty} \frac{L_{2n}L_{2n+1}}{L_{2n-1}L_{2n+2}}$$

$$= \frac{F_1}{F_2} \cdot \frac{L_2}{L_1} = \frac{1}{1} \cdot \frac{3}{1} = 3.$$

Also solved by R. André-Jeannin, Bob Prielipp, H.-J. Seiffert, and the proposer.

Trigonometric Recursion

B-674 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Define the sequence $\{u_n\}$ by

$$u_0 = 0,\ u_1 = 1,\ u_n = gu_{n-1} - u_{n-2},\ \text{for } n \text{ in } \{2, 3, \ldots\},$$

where g is a root of $x^2 - x - 1 = 0$. Compute u_n for n in $\{2, 3, 4, 5\}$ and then deduce that $(1 + \sqrt{5})/2 = 2\cos(\pi/5)$ and $(1 - \sqrt{5})/2 = 2\cos(3\pi/5)$.

Solution by Paul S. Bruckman, Edmonds, WA

Since g satisfies the equation

(1) $\quad g^2 = g + 1,$

we have

(2) $\quad g = \alpha = \frac{1}{2}(1 + \sqrt{5}) \quad$ or $\quad g = \beta = \frac{1}{2}(1 - \sqrt{5}).$

The characteristic equation of the given recurrence is

(3) $\quad z^2 - gz + 1 = 0,$

which has roots z_1 and z_2 given by

(4) $\quad z_1 = \frac{1}{2}(g + (g^2 - 4)^{1/2}), \quad z_2 = \frac{1}{2}(g - (g^2 - 4)^{1/2}).$

Making the substitution $g = 2\cos\theta$, we may express the roots in (4) as follows:

(5) $\quad z_1 = \exp(i\theta), \quad z_2 = \exp(-i\theta).$

From the initial conditions, we find that we may express u_n in the following Binet form:

(6) $\quad u_n = \dfrac{z_1^n - z_2^n}{z_1 - z_2}, \quad n = 0, 1, 2, \ldots .$

Equivalently, using (5), we obtain

(7) $\quad u_n = \dfrac{\sin n\theta}{\sin\theta}, \quad n = 0, 1, 2, \ldots .$

Using (1) and the given recurrence, we find the following values:

$$u_2 = g \cdot 1 - 0 = g; \quad u_3 = g \cdot g - 1 = g; \quad u_4 = g \cdot g - g = 1;$$

$$u_5 = g \cdot 1 - g = 0; \quad u_6 = g \cdot 0 - 1 = -1; \quad u_7 = g(-1) - 0 = -g, \text{ etc.}$$

Clearly, from (2), $g \neq \pm 2$; hence, $\theta \neq m\pi$, and $\sin\theta \neq 0$. Since

$$u_5 = \sin 5\theta / \sin\theta = 0,$$

we see that $\theta = m\pi/5$ for some integer m, not a multiple of 5. We may restrict m to the residues (mod 10), since $10\theta = 2m\pi$. Also,

$$u_2 = 2\cos\theta, \quad u_7 = \sin(2\theta + m\pi)/\sin\theta = (-1)^m \sin 2\theta/\sin\theta = (-1)^m u_2.$$

However, as we have seen, $u_7 = -u_2$; therefore, m must be odd. Moreover, since $\cos(2\pi - \theta) = \cos\theta$, we may eliminate the values $m = 7$ and 9. Therefore, $m = 1$ or 3. Then, α and β must be equal to $2\cos\pi/5$ and $2\cos 3\pi/5$, in some order. Clearly, $\alpha > 0$ and $\beta < 0$; also, $2\cos\pi/5 > 0$ and $2\cos 3\pi/5 < 0$. Therefore,

(8) $\quad \alpha = 2\cos\pi/5, \quad \beta = 2\cos 3\pi/5.$

Also solved by Herta T. Freitag, Hans Kappus, L. Kuipers, and the proposer.

Another Sine Recursion

B-675 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

In a manner analogous to that for the previous problem, show that

$$\sqrt{2 + \sqrt{2}} = 2\cos\frac{\pi}{8} \quad \text{and} \quad \sqrt{2 - \sqrt{2}} = 2\cos\frac{3\pi}{8}.$$

Solution by Paul S. Bruckman, Edmonds, WA

We have the same characteristic equation for z and the same substitutions as in B-674; however, in this case, g satisfies the equation

(1) $\quad g^4 = 4g^2 - 2.$

In this case, we may obtain the following values:

$$u_2 = g,\ u_3 = g^2 - 1,\ u_4 = g^3 - 2g,\ u_5 = g^4 - 3g^2 + 1 = g^2 - 1,$$

$$u_6 = g,\ u_7 = 1,\ u_8 = 0,\ u_9 = -1,\ u_{10} = -g, \text{ etc.}$$

As before,

(2) $\quad u_n = \sin n\theta/\sin\theta, \quad n = 0, 1, 2, \ldots,$ where $g = 2\cos\theta$.

Again, we note that $g \neq \pm 2$, so $\sin\theta \neq 0$. Since $u_8 = 0$, therefore $8\theta = m\pi$, or $\theta = m\pi/8$, for some integer m (not a multiple of 8). We see, from above, that $u_{10} = -u_2$. But

$$u_{10} = \sin(2\theta + m\pi)/\sin\theta = (-1)^m u_2;$$

hence, m must be odd. Again, we may restrict m to the residues of the period, in this case, mod 16; moreover, we may eliminate the values $m = 9$, 11, 13, and 15, since $\cos(2\pi - \theta) = \cos\theta$. Therefore, we may restrict m to the values $m = 1$, 3, 5, or 7. The roots of (1) are given by $\pm\sqrt{2 + \sqrt{2}}$ and $\pm\sqrt{2 - \sqrt{2}}$; thus, these must be equal to $2\cos m\pi/8$, $m = 1, 3, 5, 7$, in some order. Since

$$\frac{1}{2}\pi < m\pi/8 < \pi, \quad \text{for } m = 5 \text{ or } 7,$$

it is clear that the positive roots (which are the ones we are interested in) are generated by $m = 1$ or 3. Also, $\cos x$ decreases over the interval $[0, \frac{1}{2}\pi]$, from which it follows that

$$2\cos\pi/8 = \sqrt{2+\sqrt{2}}, \quad 2\cos 3\pi/8 = \sqrt{2-\sqrt{2}}.$$

Also solved by Herta T. Freitag, Hans Kappus, and the proposer.

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