A NOTE ON A CLASS OF LUCAS SEQUENCES*

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1. Introduction

In a short communication that appeared in this jounal [12], Whitford considered the generalized Fibonacci sequence $\{G_n\}$ defined as

(1.1) $\quad G_n = (\alpha_d^n - \beta_d^n)/\sqrt{d},$

where d is a positive odd integer of the form $4k + 1$ and

(1.2) $\quad \begin{cases} \alpha_d = (1 + \sqrt{d})/2 \\ \beta_d = (1 - \sqrt{d})/2. \end{cases}$

The sequence $\{G_n\}$ can also be defined by the second-order linear recurrence relation

(1.3) $\quad G_{n+2} = G_{n+1} + ((d - 1)/4)G_n; \quad G_0 = 0, \quad G_1 = 1.$

Monzingo observed [7] that, on the basis of the previous definitions, the analogous Lucas sequence $\{H_n\}$ can be defined either as

(1.4) $\quad H_{n+2} = H_{n+1} + ((d - 1)/4)H_n; \quad H_0 = 2, \quad H_1 = 1$

or, by means of the *Binet form*

(1.5) $\quad H_n = \alpha_d^n + \beta_d^n.$

Our principal aim is to extend the results established in [7] by finding further properties of the numbers H_n which, throughout this note, will be referred to as *Monzingo numbers*.

2. On the Monzingo Numbers $H_n(m)$

Letting

(2.1) $\quad (d - 1)/4 = m \in \mathbb{N}$

in (1.3) and (1.4), we have

(2.2) $\quad G_{n+2}(m) = G_{n+1}(m) + mG_n(m); \quad G_0(m) = 0, \quad G_1(m) = 1$

and the Monzingo numbers

(2.3) $\quad H_{n+2}(m) = H_{n+1}(m) + mH_n(m); \quad H_0(m) = 2, \quad H_1(m) = 1,$

respectively. Note that both $\{G_n(m)\}$ and $\{H_n(m)\}$ are particular cases of the more general sequence $\{W_n(a, b; p, q)\}$ which has been intensively studied over the past years (e.g., see [3], [4], [5], and [6]). More precisely, we have

(2.4) $\quad \{H_n(m)\} = \{W_n(2, 1; 1, -m)\}.$

The first few values of $H_n(m)$ are given in (2.5).

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$$(2.5)\quad \begin{aligned} H_0(m) &= 2\\ H_1(m) &= 1\\ H_2(m) &= (d+1)/2 = 2m+1\\ H_3(m) &= (3d+1)/4 = 3m+1\\ H_4(m) &= (d^2+6d+1)/8 = 2m^2+4m+1\\ H_5(m) &= (5d^2+10d+1)/16 = 5m^2+5m+1. \end{aligned}$$

Using Binet's form (1.5), (1.2), and the binomial theorem, the following general expression for $H_n(m)$ in terms of powers of d can readily be found to be

$$(2.6)\quad H_n(m) = H_n\left(\frac{d-1}{4}\right) = \frac{1}{2^{n-1}}\sum_{j=0}^{[n/2]}\binom{n}{2j}d^j,$$

where $[\cdot]$ denotes the greatest integer function.

From (2.3), it must be noted that $H_n(1)$ and the n^{th} Lucas numbers L_n coincide. As a special case, letting $m = 1$ (i.e., $d = 5$) in (2.6), we obtain

$$(2.7)\quad L_n = \frac{1}{2^{n-1}}\sum_{j=0}^{[n/2]}\binom{n}{2j}5^j.$$

Countless identities involving the numbers $H_n(m)$ and $G_n(m)$ can be found with the aid of (1.1) and (1.5). A few examples of the various types are listed below.

$$(2.8)\quad H_n(m)H_{n+k}(m) = H_{2n+k}(m) + (-m)^n H_k(m) \quad \text{(cf. [7, (3)])},$$

whence *Simson's formula* for $\{H_n(m)\}$ turns out to be

$$(2.9)\quad H_{n-1}(m)H_{n+1}(m) - H_n^2(m) = (-m)^{n-1}(4m+1).$$

$$(2.10)\quad H_{2n}(m) = H_n^2(m) - 2(-m)^n,$$

$$(2.11)\quad G_{2n}(m) = G_n(m)H_n(m),$$

$$(2.12)\quad H_{2n+1}(m) = \frac{(4m+1)G_{2n}(m) + H_{2n}(m)}{2},$$

$$(2.13)\quad G_{2n+1}(m) = \frac{G_{2n}(m) + H_{2n}(m)}{2},$$

$$(2.14)\quad \sum_{j=0}^{n} H_{aj+b}(m) = \frac{(-m)^a(H_{an+b}(m) - H_{b-a}(m)) - H_{a(n+1)+b}(m) + H_b(m)}{(-m)^a + 1 - H_a(m)}.$$

Observe that (2.14) may involve the use of the negative-subscripted Monzingo numbers

$$(2.15)\quad H_{-n}(m) = (-m)^{-n}H_n(m).$$

$$(2.16)\quad C^{(1)} = \sum_{j=1}^{n} H_j(m)H_{n-j+1}(m) = \frac{[2m(2n-1)+n]H_{n+1}(m) - 2m^2H_{n-1}(m)}{4m+1},$$

($\{C_n^{(1)}\}$, the *Monzingo 1st Convolution Sequence*)

$$(2.17)\quad \sum_{j=1}^{n} jH_j(m) = \frac{nH_{n+4}(m) - (n+1)H_{n+3}(m) + 3m+1}{m^2},$$

$$(2.18)\quad \sum_{j=0}^{\infty}\frac{H_j(m)}{j!} = \exp\left(\frac{1+\sqrt{4m+1}}{2}\right) + \exp\left(\frac{1-\sqrt{4m+1}}{2}\right).$$

The usefulness of (2.10)-(2.13) will be explained later.

Some properties of the Monzingo numbers can also be found by using appropriate matrices. As a minor example, we invite the reader to prove that

(2.19) $H_n(m) = \operatorname{tr} M^n$,

where tr A denotes the trace (sum of diagonal entries) of a generic square matrix A and

(2.20) $M = \begin{bmatrix} 1 & m \\ 1 & 0 \end{bmatrix}$.

Letting

(2.21) $m = k(k + 1) \quad (k \in \mathbb{N})$

in (2.3) leads to a simple but rather interesting case. In fact, we have [cf. (2.1)]

(2.22) $d = 4k^2 + 4k + 1 = (2k + 1)^2$,

so that [cf. (1.2)]

(2.23) $\alpha_d = k + 1 \quad$ and $\quad \beta_d = -k$

are integral and

(2.24) $H_n(k^2 + k) = (k + 1)^n + (-k)^n$.

On the basis of (2.24), it can readily be seen that the numbers $H_n(k^2 + k)$ can be expressed by means of the following first-order linear recurrence relation

(2.25) $H_n(k^2 + k) = (k + 1)H_{n-1}(k^2 + k) + (2k + 1)k^{n-1}(-1)^n$;
$H_0(k^2 + k) = 2$.

This suggests an analogous expression for $H_n(m)$ (m arbitrary). In fact, using (1.2), (1.5), and (2.1), it can be proved that

(2.26) $H_n(m) = \alpha_d H_{n-1}(m) - \sqrt{4m + 1}\,\beta_d^{n-1}; \quad H_0(m) = 2$,

whence, as a special case, we have

(2.27) $L_n = \alpha L_{n-1} - 5\beta^{n-1}; \quad L_0 = 2$,

where $\alpha = \alpha_5$ and $\beta = \beta_5$.

Now, let us consider a well-known (e.g., see [6], Cor. 7) divisibility property of the numbers $W_n(2, b; b, q)$ which, obviously, applies to the Monzingo numbers. Namely, we can write

(2.28) $H_r(m) \,|\, H_{r(2s+1)}(m)$

whence it follows that

Proposition 1: If $H_n(m)$ is a prime, then n is either a prime or a power of 2.

Proposition 1 and (2.24) give an alternative proof of a particular case (a and b, consecutive integers) of well-known number-theoretic statements concerning the divisors of $a^n \pm b^n$ (e.g., see [10], pp. 184ff.). More precisely, we can state

Proposition 2 (n odd): If $(k + 1)^n - k^n$ is a prime, then n is a prime.

Proposition 3 (n even): If $(k + 1)^n + k^n$ is a prime, then $n = 2^h$ ($h \in \mathbb{N}$).

It must be noted that, for $k = 1$, Proposition 2 is the well-known Mersenne's theorem, while Proposition 3 is related to a property concerning Fermat's numbers (e.g., see [10], p. 107). We point out that, from the said statements concerning the factors of $a^n \pm b^n$, it follows that, if p is an odd prime and

$H_p(k^2 + k)$ is composite, then its prime factors are of the form $2lp + 1$. For example, we can readily check that, for $k = 2$ and $p = 11$, we have

$$H_{11}(6) = 175099 = (2 \cdot 1 \cdot 11 + 1)^2(2 \cdot 15 \cdot 11 + 1).$$

Finally, let us consider the sum

$$S_{n,h} = \sum_{m=0}^{h} H_n(m) \tag{2.29}$$

and ask ourselves whether it is possible to find a closed form expression for (2.29) in terms of powers of h. A modest attempt in this direction is shown below. Taking into account that $H_n(0) = 1 \ \forall \ n > 0$, expressions valid for the first few values of n can easily be derived from (2.5) and from the calculation of $H_6(m) = 2m^3 + 9m^2 + 6m + 1$:

$$\begin{aligned} &S_{1,h} = h + 1 && S_{4,h} = (2h^3 + 9h^2 + 10h + 3)/3 \\ &S_{2,h} = h^2 + 2h + 1 && S_{5,h} = (5h^3 + 15h^2 + 13h + 3)/3 \\ &S_{3,h} = (3h^2 + 5h + 2)/2 && S_{6,h} = (h^4 + 8h^3 + 16h^2 + 11h + 2)/2. \end{aligned} \tag{2.30}$$

3. Some Congruence and Divisibility Properties of the Monzingo Numbers

If we rewrite (2.6) as

$$2^{n-1}H_n(m) = 1 + \sum_{j=1}^{[n/2]} \binom{n}{2j} d^j, \tag{3.1}$$

it is easily seen that

$$2^{n-1}H_n(m) \equiv 1 \pmod{d}. \tag{3.2}$$

From (2.24), Proposition 1, and the definition of *perfect numbers* (e.g., see [9], p. 81), it follows that all even perfect numbers are given by $2^{p-1}H_p(2)$, where $H_p(2)$ is prime ($p \geq 3$, a prime). Since $m = 2$ implies $d = 9$, from (3.2) we can state

Proposition 4: Any even perfect number greater than 6 is congruent to 1 modulo 9.

By using either [1, (2)] or [2, (1.2)] and taking into account that [cf. (1.2)]

$$\begin{cases} \alpha_d + \beta_d = 1 \\ \alpha_d \beta_d = (1 - d)/4 = -m, \end{cases} \tag{3.3}$$

we obtain the following expression for $H_n(m)$ in terms of powers of m [cf. (2.5)]

$$H_n(m) = \sum_{j=0}^{[n/2]} nC_{n,j} m^j \quad (n \geq 1), \tag{3.4}$$

where

$$C_{n,j} = \frac{1}{n-j}\binom{n-j}{j}. \tag{3.5}$$

Rewrite (3.4) as

$$H_n(m) = 1 + n \sum_{j=1}^{[n/2]} C_{n,j} m^j \quad (n \geq 1) \tag{3.6}$$

and observe that, if n is a prime, then $C_{n,j}$ is integral. It follows that

$$H_n(m) \equiv 1 \pmod{n} \text{ if } n \text{ is prime.} \tag{3.7}$$

Note that (3.6) allows us to state that

(3.8) (i) $H_n(m) \equiv 1 \pmod{m} \quad (n \geq 1)$

(3.9) (ii) $H_n(2k)$ is odd $(n \geq 1)$,

(3.10) (iii) $H_n(2k+1) \equiv 1 + \sum_{j=1}^{[n/2]} nC_{n,j} = L_n \pmod{2}$,

(3.10′) that is to say, $H\ (2k+1)$ is even iff $n \equiv 0 \pmod{3}$.

Curiosity led us to investigate the divisibility of $H_n(m)$ by some primes $p > 2$. A computer experiment was carried out to determine the necessary and sufficient conditions on n for an odd prime $p \leq 47$ to be a divisor of $H_n(m)$ $(2 \leq m \leq 10)$. The case $m = 1$ has been disregarded, since the conditions on n for the congruence $L_n \equiv 0 \pmod{p}$ $(p \leq 47)$ to hold are well known. For p and m varying within the above said intervals, the results can be summarized as follows

(3.11) $H_n(m) \equiv 0 \pmod{p}$ iff $n \equiv r \pmod{2r}$.

The values of r are displayed in Table 1, where a blank value denotes that p is not a divisor of the Monzingo sequence $\{H_n(m)\}$.

TABLE 1. Values of r for $3 \leq p \leq 47$ and $2 \leq m \leq 10$

$p \backslash m$	2	3	4	5	6	7	8	9	10
3	-	-	2	-	-	2	-	-	2
5	2	3	-	-	-	2	3	-	-
7	3	2	4	-	-	-	4	3	2
11	-	6	6	2	-	3	-	5	-
13	6	-	3	7	2	6	7	-	-
17	4	8	-	8	8	9	2	-	9
19	-	-	9	10	3	10	5	2	5
23	11	11	12	6	11	-	4	12	-
29	14	14	-	-	7	-	14	-	5
31	5	4	16	16	-	16	15	16	3
37	18	-	-	18	18	-	18	-	-
41	10	21	-	-	20	21	10	5	-
43	-	-	21	-	21	-	-	22	7
47	23	8	23	-	23	8	24	23	6

Let us give an example of use of Table 1 by considering the case $m = 6$ and $p = 29$. For these two values, the table gives $r = 7$. It means that $H_n(6) \equiv 0 \pmod{29}$ iff $n \equiv 7 \pmod{14}$.

Of course, the above-mentioned experiment led us to discover also the repetition period $P_{m,p}$ of the Monzingo sequences reduced modulo p. Some values of $P_{m,p}$ are shown in Table 2.

TABLE 2. Values of $P_{m,p}$ for $3 \leq p \leq 47$ and $2 \leq m \leq 10$

$p \backslash m$	2	3	4	5	6	7	8	9	10
3	2	1	8	2	1	8	2	1	8
5	4	24	6	1	4	4	24	6	1
7	6	24	48	3	6	1	16	6	24
11	10	120	120	40	5	60	10	10	6
13	12	12	12	56	12	12	56	84	42
17	8	16	8	16	16	288	16	144	288
19	18	90	18	360	18	120	60	72	180
23	22	22	528	264	22	11	176	528	11
29	28	28	35	105	28	28	28	210	280
31	10	240	320	192	30	960	30	960	30
37	36	171	171	36	36	684	36	36	36
41	20	336	105	40	40	1680	20	40	20
43	14	42	42	42	42	33	77	1848	42
47	46	736	46	23	46	736	2208	46	552

3.1 The Numbers $H_n^{(1)}(m)$: A Divisibility Property

Both the definitions and most of the properties of the numbers $H_n(m)$ and $G_n(m)$ remain valid if m is an arbitrary (not necessarily integral) quantity. Let us define the numbers $H_n^{(1)}(m)$ as the first derivative of $H_n(m)$ with respect to m

$$(3.12)\quad H_n^{(1)}(m) = \frac{d}{dm} H_n(m).$$

From (3.12) and (3.4), we have

$$(3.13)\quad H^{(1)}(m) = \sum_{j=0}^{[n/2]} j \frac{n}{n-j}\binom{n-j}{j} m^{j-1} = \sum_{j=1}^{[n/2]} n \frac{(n-j-1)!}{(j-1)!(n-2j)!} m^{j-1}$$
$$= n \sum_{j=1}^{[n/2]} \binom{n-j-1}{j-1} m^{j-1} \quad (n \ge 1).$$

Now it is plain that $H_n^{(1)}(m) \equiv 0 \pmod{n}$. Moreover (cf. [6], p. 278), (3.13) leads to the following cute result

$$(3.14)\quad \frac{H_n^{(1)}(m)}{n} = G_{n-1}(m) \quad (n \ge 1).$$

4. The Monzingo Pseudoprimes

Of course, the converse of (3.7) is not always true. Let us define the odd composites satisfying (3.7) as *Monzingo Pseudoprimes of the m^{th} kind* and abbreviate them *m-M.Psps*. Incidentally, we note that the 1-M.Psps. and the Fibonacci pseudoprimes defined in [8] and investigated in [2] coincide.

For $m > 1$, the m-M.Psps. are not as rare as the Fibonacci pseudoprimes. Let $\mu_m(x)$ be the m-M.Psp.-counting function (i.e., the number of m-M.Psps. not exceeding x) and let $M_1(m)$ be the smallest among them. A computer experiment has been carried out to obtain $\mu_m(1000)$ and $M_1(m)$ for $1 \le m \le 25$. These quantities are shown against m in Tables 3 and 4, respectively.

TABLE 3. Values of $\mu_m(1000)$ for $1 \le m \le 25$

m	$\mu_m(1000)$	m	$\mu_m(1000)$
1	1	14	11
2	3	15	22
3	6	16	2
4	5	17	5
5	8	18	8
6	15	19	13
7	9	20	17
8	3	21	29
9	15	22	9
10	14	23	4
11	7	24	10
12	15	25	9
13	12		

TABLE 4. Values of $M_1(m)$ for $1 \le m \le 25$

m	$M_1(m)$	m	$M_1(m)$
1	705	14	21
2	341	15	9
3	9	16	85
4	25	17	51
5	15	18	9
6	9	19	25
7	49	20	15
8	231	21	9
9	9	22	33
10	25	23	69
11	33	24	9
12	9	25	25
13	49		

The reader who would enjoy discovering many more m-M.Psps. can use the simple computer algorithm described on pages 239-40 of [2], after replacing the identities (3.5)-(3-8) in [2] by the identities (2.10)-(2.13) shown in Section 2 above.

It can be proved that certain odd composites are m-M.Psps. In this note, we restrict ourselves to demonstrating that, for p an odd prime and s an integer greater than 1, p^s is a p-M.Psp.

Theorem 1: $H_{p^s}(p) \equiv 1 \pmod{p^s}$.

Proof: By observing (3.6), it is plain that it suffices to prove that $C_{p^s,j}\,p^j$ is integral for $1 \le j \le (p^s - 1)/2$. More precisely [cf. (3.5)], if $(p, j) = 1$, then $C_{p^s,j}$ is an integer; thus, it suffices to prove that the power a with which p enters into $j!$ is less than j. This is true for any j and p (odd). In fact, it is known (e.g., see [11], p. 21) that

$$(4.1) \qquad a = \sum_{i=1}^{\infty} [j/p^i],$$

whence we can write

$$a < \sum_{i=1}^{\infty} j/p^i = j/(p-1) < j. \quad \text{Q.E.D.}$$

Let us conclude this note by pointing out that the numerical evidence turning out from the above said computer experiment suggests the following

Conjecture 1: If $p \ge 5$ is a prime and s is an integer greater than 1, then p^s is a $(p-1)$-M.Psp., that is

$$(4.2) \qquad H_{p^s}(p-1) \equiv 1 \pmod{p^s}.$$

For some values of p, we checked Conjecture 1 by ascertaining that, while the addends $C_{p^s,j}\,(p-1)^j$ are in general *not* integral, the sum

$$(4.3) \qquad \sum_{j=1}^{(p^s-1)/2} C_{p^s,j}(p-1)^j$$

is. For example, let us consider the case $p = 7$, $s = 2$ and show that (4.3) is integral. The nonintegral addends in (4.3) are those for which g.c.d.$(p^s - j, j) \ne 1$, that is

$$(4.4) \qquad A_1 = \frac{1}{42}\binom{42}{7}6^7, \quad A_2 = \frac{1}{35}\binom{35}{14}6^{14}, \quad A_3 = \frac{1}{28}\binom{28}{21}6^{21}.$$

Let us write

$$(4.5) \qquad A_1 + A_2 + A_3 = \frac{41 \cdot 39 \cdot 38 \cdot 37 \cdot 2}{7}6^7 + \frac{34 \cdot 31 \cdot 29 \cdot 23 \cdot 11 \cdot 5 \cdot 4 \cdot 3}{7}6^{14} + \frac{26 \cdot 23 \cdot 11 \cdot 9 \cdot 5}{7}6^{21}$$

and reduce the sum of the numerators on the right-hand side of (4.5) modulo 7

$$6 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \cdot 6 + 6 \cdot 3 \cdot 1 \cdot 2 \cdot 4 \cdot 5 \cdot 4 \cdot 3 \cdot 1 + 5 \cdot 2 \cdot 4 \cdot 2 \cdot 5 \cdot 6$$
$$\equiv 6 + 2 + 6 \equiv 0 \pmod 7.$$

It follows that $A_1 + A_2 + A_3$ is integral, so that 49 is a 6-M.Psp.

References

1. O. Brugia & P. Filipponi. "Waring Formulae and Certain Combinatorial Identities." Fond. U. Bordoni Techn. Rept. 3B5986 (1986).
2. A. Di Porto & P. Filipponi. "More on the Fibonacci Pseudoprimes." *Fibonacci Quarterly* *27.3* (1989):232-42.
3. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *Fibonacci Quarterly* *3.3* (1965):161-76.

4. A. F. Horadam. "Special Properties of the Sequence $W_n(a, b; p, q)$." *Fibonacci Quarterly* 5.5 (1967):424-34.
5. A. F. Horadam. "Pell Identities." *Fibonacci Quarterly* 9.3 (1971):245-52, 263.
6. Jin-Zai Lee & Jia-Sheng Lee. "Some Properties of the Sequence $\{W_n(a, b; p, q)\}$." *Fibonacci Quarterly* 25.3 (1987):268-78, 283.
7. M. G. Monzingo. "An Observation Concerning Whitford's 'Binet's Formula Generalized.'" *A Collection of Manuscripts Related to the Fibonacci Sequence*, pp. 93-94. Edited by V. E. Hoggatt, Jr., & M. Bicknell-Johnson. Santa Clara: The Fibonacci Association, 1980.
8. J. M. Pollin & I. J. Schoenberg. "On the Matrix Approach to Fibonacci Numbers and the Fibonacci Pseudoprimes." *Fibonacci Quarterly* 18.3 (1980):261-68.
9. P. Ribenboim. *The Book of Prime Number Records.* New York: Springer-Verlag, 1988.
10. H. Riesel. *Prime Numbers and Computer Methods for Factorization.* Boston: Birkäuser Inc., 1985.
11. I. M. Vinogradov. *Elements of Number Theory*. New York: Dover, 1954.
12. A. K. Whitford. "Binet's Formula Generalized." *Fibonacci Quarterly* 15.1 (1977):21.

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