CONJECTURES ABOUT s-ADDITIVE SEQUENCES

Steven R. Finch
6 Foster Street, Wakefield, MA 01880
(Submitted July 1989)

A strictly increasing sequence of positive integers a_1, a_2, ... is defined to be *s-additive* [1] if, for $n > 2s$, a_n is the least integer greater than a_{n-1} having precisely s representations $a_i + a_j = a_n$, $i < j$. The first $2s$ terms of an s-additive sequence are called the *base* of the sequence. An s-additive sequence, for a given base, may be either finite or infinite; the sequence is assumed to be maximal in the sense that the total number of terms is as large as possible. Consider, for example, the case in which $s = 1$, $a_1 = 1$, and $a_2 = 2$. The next fifteen terms of the sequence are 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53. The sequence is infinite (as is any 1-additive sequence) since $a_{n-3} + a_{n-1}$ is an integer greater than a_{n-1} with no other representation $a_i + a_j$ and, hence, there exists a least such integer. It is the archetypal s-additive sequence, and was first studied by Stanislaw Ulam [2]. An example of a 2-additive sequence is 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 17, 19, 29, 31, 33, 43, 44, 47, 51, ..., which also appears to be infinite (though a proof of this fact is not known). Not all 2-additive sequences are infinite, as illustrated by the sequence 1, 3, 5, 7, 8.

For $s \geq 1$, Raymond Queneau [1] showed that an s-additive sequence has at least $2s + 2$ terms if and only if there exist positive integers u and v such that the $2s$ numbers in the base (up to ordering) are u, $2u$, ..., su, v, $u + v$, $2u + v$, ..., $(s - 1)u + v$. This is called *Condition* u, v. We denote an s-additive sequence satisfying Condition u, v by the ordered triple (s, u, v). Note that the correspondence between such sequences and ordered triples is not one-to-one, since $(s, 1, s + 1) = (s, 2, 1)$. Queneau undertook a detailed examination of various properties of s-additive sequences, including conditions for sequences to be infinite and conditions for sequences to be regular (in a sense to be defined shortly). Some of the conjectures in [1] are consistent with conjectures presented here; some others are false due to several unfortunate errors in Queneau's computations.

We examine first conditions for s-additive sequences to be infinite.

Conjecture 1: A 2-additive sequence is infinite if and only if Condition u, v is met.

For $s \geq 3$, Condition u, v is necessary but not sufficient for infinitude, as evidenced by the finite 4-additive sequence $(4, 1, 5)$ = 1 (1) 10, 12 (2) 20, 23 (2) 31, 36, 38, 47, 48, 49, 51, 53, 60, 80, 85. In order to state Conjectures 2 through 4, we assume that Condition u, v is satisfied and that, without loss of generality, u and v are relatively prime. These two assumptions hold throughout the remainder of this paper.

Conjecture 2: An s-additive sequence, when $3 \leq s \leq 6$, is infinite if and only if

(a) $u = 1$ and v is as in Table 1,
(b) $u = 2$ and v is as in Table 2, or
(c) $u \geq 3$.

Conjecture 3: An s-additive sequence, when s is even and $8 \leq s \leq 20$, is infinite if and only if

(a) $u = 2$ and v is as in Table 3, or
(b) $u \geq 3$.

Table 1: Conditions associated with Conjecture 2(a)

s	Conditions on v
3	$v > 3$
4	$5 < v \neq 8, 12, 13, 17, 22$
5	$v = 6, 9, 13, 15$
6	$v = 8$

Table 2: Conditions associated with Conjecture 2(b)

s	Conditions on v
3	$v = 1, 5$ or $\equiv 3 \bmod 4$
4	$v \equiv 3, 7, 9 \bmod 10$
5	$v = 1$ or $\equiv 9 \bmod 12$
6	$v \equiv 3, 5, 7, 9, 11 \bmod 14$

Table 3: Conditions associated with Conjecture 3(a)

s	Conditions on v
8	$v = 3, 11, 21, 25, 29, 39, 57, 61, 65, 75, 83, 93, 97, 101, 111, 119, 129, 133, 137, 147$ or $151 < v \equiv 3, 7, 11 \bmod 18$
10	$v = 19, 23, 45, 51, 67, 89, 95, 107$ or $111 < v \equiv 1, 7, 19 \bmod 22$
12	$v = 47, 143, 169, 177, 183, 235, 261, 307, 313, 333, 339, 365, 391$ or $411 < v \equiv 1, 21 \bmod 26$
14	$v = 189, 249, 279, 309, 339, 369, 375, 399, 429, 459, 489, 519, 525, \ldots, 939$ or $945 < v \equiv 9, 15 \bmod 30$
16	$v = 187, 323, 663, 731, 833, 893, 935, 969, 995, 1003, 1029, 1037, 1063, \ldots, 1649$ or $1675 < v \equiv 9, 17 \bmod 34$
18	$v = 417, 645, 759, 873, 979, 987, 1101, 1215, 1329, 1443, 1519, 1557, \ldots, 3305$ or $3343 < v \equiv 37 \bmod 38$
20	$v = 439, 1333, 1343, 1543, 1573, 1615, 1627, 1637, 1657, 1699, 1741, 1783, 1867, 1889, \ldots, 4429$ or $4451 < v \equiv 19, 41 \bmod 42$

Conjecture 4: An s-additive sequence, when s is odd and $s \geq 7$, is infinite if and only if $u \geq 3$.

The sequence (24, 2, 1523) appears to be infinite, whereas (22, 2, v) is never infinite. Proof that certain sequences are finite is not difficult; for example, (3, 2, v) has $(7v + 53)/4$ terms ($a_{(7v+53)/4} = 10v + 10$) when $5 < v \equiv 1 \bmod 4$. However, no s-additive sequence, $s > 1$ and $u \leq 2$, has been proven to be infinite. Note that the example involving (3, 2, v) shows that arbitrarily long finite sequences exist. Long sequences are computationally unwieldy since all terms $a_1, \ldots, a_{n-1}$ must be considered when determining a_n. Thus, the computer evidence leading to Conjectures 1 through 4 is necessarily limited.

We turn now to regularity issues. An infinite s-additive sequence is *regular* if successive differences $a_{n+1} - a_n$ are eventually periodic; i.e., there is a positive integer N such that $a_{N+n+1} - a_{N+n} = a_{n+1} - a_n$ for all sufficiently large n. (The smallest such N is called the *period*.) An equivalent condition involves *arithmetic multiprogressions* [1] which are infinite sequences of the form

$$c_1, c_2, \ldots, c_k, b + c_1, b + c_2, \ldots, b + c_k,$$
$$2b + c_1, 2b + c_2, \ldots, 2b + c_k, \ldots,$$

where $0 < c_1 < c_2 < \cdots < c_k < b + c_1$. If some arithmetic multiprogression, after at most finitely many deletions of certain terms or insertions of additional terms, is equal to the s-additive sequence (s, u, v), then (s, u, v) is regular. We write this condition more compactly as

$$(s, u, v) \sim bn + [c_1, c_2, \ldots, c_k] \quad n = 0, 1, 2, \ldots,$$

in which the symbol ~ is to be interpreted as eventual equality. If greater precision is required, we write

$$(s, u, v) = bn + [c_1, c_2, \ldots, c_k] \oplus d_1, \ldots, d_p \ominus e_1, \ldots, e_q,$$

where $d_1, \ldots, d_p$ and $e_1, \ldots, e_q$ are, respectively, the inserted and deleted terms on the right-hand side that make equality hold.

The nature of Conjectures 2 through 4 might lead one to suspect that something is special about the case $u \geq 3$. This is true, in fact, as proved by Queneau in [1].

Theorem 1: If $s > 1$ and $u \geq 3$, then (s, u, v) is regular and

$$(s, u, v) = nu + [v] \oplus u, 2u, \ldots, su, (2s - 1)u + 2v.$$

A consequence of this result and Conjecture 4 is that there do not exist infinite irregular sequences when s is odd and $s \geq 7$. No analogous general formulas appear to hold for the remaining cases $s = 1$ or $u \leq 2$. A limited computer search for regular 1-additive sequences has uncovered many examples, some of which are exhibited in Table 4. (The first three of these were found by Queneau [1].) We conjecture that $(1, u, v)$ is regular for a wide variety of u and v. Though a proof is not known, a sensible argument might be based on Theorem 2 and (deceptively simple) Conjecture 5. Periods for $(1, 2, v)$ and for $(1, 4, v)$, as fascinatingly intricate functions of odd $v > 3$, are listed in Table 5. [Some cases have either incalculably long periods or long initial stretches before periodicity begins. For example, the period for $(1, 2, v)$, where $35 \leq v \leq 41$ is odd, probably exceeds 10^9.]

Table 4: Regular 1-additive sequences

$(1, 2, 5) = 126n + [5\ (2)\ 15,\ a_9 = 19,\ a_{10} = 23, \ldots, a_{34} = 119] \oplus 2, 12$

$(1, 2, 7) = 126n + [7\ (2)\ 21,\ a_{11} = 25,\ a_{12} = 29, \ldots, a_{28} = 117] \oplus 2, 16$

$(1, 2, 9) = 1778n + [9\ (2)\ 27,\ a_{13} = 31,\ a_{14} = 35, \ldots, a_{446} = 1767] \oplus 2, 20$

$(1, 2, 11) = 6510n + [11\ (2)\ 33,\ a_{15} = 37,\ a_{16} = 41, \ldots, a_{1630} = 6497] \oplus 2, 24$

$(1, 2, 13) = 23622n + [13\ (2)\ 39,\ a_{17} = 43,\ a_{18} = 47, \ldots, a_{5908} = 23607] \oplus 2, 28$

$(1, 2, 15) = 510n + [15\ (2)\ 45,\ a_{19} = 49,\ a_{20} = 53, \ldots, a_{82} = 493] \oplus 2, 32$

$(1, 2, 17) = 507842n + [17\ (2)\ 51,\ a_{21} = 55,\ a_{22} = 59, \ldots, a_{126962} = 507823] \oplus 2, 36$

$(1, 4, 5) = 192n + [5\ (4)\ 17, 19, 21,\ a_{10} = 25,\ a_{11} = 27, \ldots, a_{35} = 173] \oplus 4, 14, 24$

$(1, 4, 9) = 640n + [9\ (4)\ 29, 31, 33, 37\ (2)\ 41,\ a_{15} = 45,\ a_{16} = 47, \ldots, a_{91} = 609] \oplus 4, 22, 40$

$(1, 4, 11) = 1318n + [11\ (4)\ 27, 37, 39, 43\ (2)\ 47, 51\ (2)\ 57, 61, 67, 69, 75, 77, 83, 85, 89, 91, 99, 105,\ a_{29} = 111,\ a_{30} = 123, \ldots, a_{249} = 1309] \oplus 4, 26, 31, 35, 48 \ominus 57, 105$

$(1, 4, 13) = 896n + [13\ (4)\ 41, 43, 45, 49\ (2)\ 53,\ a_{17} = 57,\ a_{18} = 59, \ldots, a_{107} = 853] \oplus 4, 30, 56$

$(1, 4, 17) = 2304n + [17\ (4)\ 53, 55, 57, 61\ (2)\ 65, 69\ (2)\ 73,\ a_{22} = 77,\ a_{23} = 79, \ldots, a_{251} = 2249] \oplus 4, 38, 72$

$(1, 4, 19) \sim 2560n + [a_{2552} = 14753,\ a_{2553} = 14761, \ldots, a_{2903} = 17275]$

$(1, 4, 21) = 2816n + [21\ (4)\ 65, 67, 69, 73\ (2)\ 77, 81\ (2)\ 85,\ a_{24} = 89,\ a_{25} = 91, \ldots, a_{283} = 2749] \oplus 4, 46, 88$

Table 5: Periods for $(1, u, v)$, $u = 2$ and 4

v	$u = 2$	$u = 4$
5	32	32
7	26	-
9	444	88
11	1628	246
13	5906	104
15	80	-
17	126960	248
19	380882	352
21	2097152	280
23	1047588	5173
25	148814	304
27	8951040	10270
29	5406720	320
31	242	-
33	127842440	712
35	-	826
37	-	776
39	-	108966
41	-	824

Theorem 2: If a 1-additive sequence has only finitely many even terms, then the sequence is regular.

Proof; Let e denote the number of even terms in the 1-additive sequence a_1, a_2, a_3, ... Let $x_1 < x_2 < \cdots < x_e$ be the even terms and let $y_k = x_k/2$ for each k, where $1 \le k \le e$. Given an integer $n \ge y_e$, define

$$b_n = \text{the number of representations } a_i + a_j = 2n + 1,\ i < j.$$

Observe that $a_i + a_j = 2n + 1$ only if either a_i or a_j is equal to some x_k (since a sum of two integers is odd if and only if one of the integers is odd and the other is even). This observation gives rise to the following recursive formula:

$$b_n = \sum_{k=1}^{e} \delta(b_{n-y_k} - 1)$$

where $\delta(0) = 1$ and $\delta(r) = 0$ for $r \ne 0$. The summation simply counts the number of times (out of e) that $2n - x_k + 1$ is a term in a_1, a_2, ... Define now, for each $n \ge x_e$, a vector of y_e components

$$\beta_n = (b_{n-y_e}\ b_{n-y_e+1}\ b_{n-y_e+2}\ \cdots\ b_{n-1})^{\mathsf{T}}.$$

Regularity of the 1-additive sequence a_1, a_2, ... is clearly equivalent to eventual periodicity of the vector sequence β_{x_e}, β_{x_e+1}, ... The components of β_n obviously do not exceed e. Since the number of vectors of length y_e containing 0, 1, ..., $e - 1$ or e is $(e + 1)^{y_e} < \infty$, some β_n must recur, which, in turn, brings about periodicity by the recursive formula. This completes the proof.

Recall that u and v are assumed to be relatively prime. Assume, moreover, that $u < v$.

Conjecture 5:

$(1, 1, v)$ has infinitely many even terms.
$(1, 2, v)$ has two even terms (specifically $a_1 = 2$ and $a_{(v+7)/2} = 2v + 2$) when $v > 3$; it has infinitely many even terms when $v = 3$.
$(1, 3, v)$ has infinitely many even terms.

$(1, 4, v)$ has four even terms when $v = 2^k - 1$ for some $k = 3, 4, 5, \ldots$; otherwise, it has three even terms.
$(1, 5, v)$ has thirteen even terms when $v = 6$; otherwise, it has infinitely many even terms.
$(1, u, v)$, for even $u \geq 6$, has $2 + u/2$ even terms.
$(1, u, v)$, for odd $u \geq 7$, has $2 + v/2$ even terms when v is even; otherwise, it has infinitely many even terms.

There is no reason for even terms to be small; for example, $(1, 4, 255)$ has $a_{8750} = 260606$.

Other interesting trends exist in the distribution of successive differences $a_{n+1} - a_n$ for these sequences. Let us focus on $(1, 2, v)$, $v > 3$, for definiteness. The successive differences are always even beyond a certain point. For most of a period, the successive differences remain relatively small. As the end of the period draws near, the successive differences seem to explode to a maximum value ($= 2v + 2$), which concludes the period and a new period begins. In contrast, the sequence $(1, 2, 3)$ appears to possess unbounded successive differences. This seems to occur as well for the sequence $(s, 1, s + 1)$, for each $s = 1, 2, 3$, and 5; e.g., when $s = 2$, $a_{9384} - a_{9383} = 174886 - 174579 = 307$. Many questions arise. Is the converse of Theorem 2 true? Do there exist regular s-additive sequences for $s > 1$ and $u \leq 2$? Is it possible for successive differences of an infinite *irregular* s-additive sequence to be *bounded*?

Queneau also introduces several generalizations of s-additivity, of which we discuss one. (Replacing addition by multiplication in the definition of s-additivity defines s-multiplicativity. This has not been studied. Nor has substituting the condition $i < j$ by $i \leq j$.) A strictly increasing sequence of positive integers $a_1, a_2, \ldots$ is defined to be (s, t)-*additive with base* B if B consists of the first m terms $a_1, a_2, \ldots, a_m$ for some positive integer m and if, for $n > m$, a_n is the least integer greater than a_{n-1} having precisely s representations of the form

$$a_{i_1} + a_{i_2} + \cdots + a_{i_t} = a_n, \quad i_1 < i_2 < \cdots < i_t.$$

Note that an s-additive sequence is the same as an $(s, 2)$-additive sequence with $m = 2s$. Note also that, while $m \geq 2s$ is necessary for $(s, 2)$-additivity and $m \geq t$ is necessary for $(1, t)$-additivity, $m = 5$ is possible in conjunction with $(2, 3)$-additivity. Lacking a suitable analogue of Condition u, v for s-additivity, we write an (s, t)-additive sequence as $(s, t; a_1, \ldots, a_m)$. For example,

$(2,3;1,2,3,4,5) = 1(1)5, 8(1)11, 25, 28, 29, 49, 66, 67, 69, 89, 92, 110, 111, \ldots$

which appears to be infinite. As previously, any $(1, t)$-additive sequence, for $t \geq 2$, is infinite, while extension of the proof to (s, t)-additive sequences, for $s > 1$, does not seem possible. We conclude with several more arithmetic multiprogression formulas obtained by limited computer search for regular $(1, 3)$-additive sequences (see Table 6). The first of these was found by Peter N. Muller and also appears in [3].

Table 6: Regular (1, 3)-additive sequences

$(1, 3; 1, 2, 3)$	~	$25n$ + [80, 82, 104]
$(1, 3; 1, 2, 9)$	~	$572n$ + [581 (1) 590, 645 (1) 653, 708 (1) 717, 772 (1) 781, 836 (1) 844, 899 (1) 908, 963 (1) 972, 1027 (1) 1035, 1090 (1) 1098]
$(1, 3; 1, 3, 4)$	~	$219n$ + [411, 412, 444, 446, 481, 482, 517, 521, 554, 555, 591, 626]
$(1, 3; 1, 3, 5)$	~	$82n$ + [87, 89, 115, 117, 141, 143]
$(1, 3; 1, 3, 6)$	~	$51n$ + [164 (1) 167, 211 (1) 213]
$(1, 3; 1, 3, 7)$	~	$20n$ + [23]
$(1, 3; 2, 3, 4)$	~	$148n$ + [157, 159, 160, 203, 204, 206 (1) 208, 253 (1) 255, 258, 302]

Acknowledgment

I wish to thank Jane Hale [4] for introducing me to Queneau's literary work and for bringing my attention to the sequences discussed in this paper.

Postscript

Recent computations show that

$$(1, 4, 7) \sim 11301098n + [a_{13671499} = 80188457, \ldots, a_{15599457} = 91489549]$$

and

$$(1, 5, 6) \sim 1720n + [a_{156303} = 1579049, \ldots, a_{156510} = 1580767];$$

thus, (1, 4, 7) and (1, 5, 6) have periods 1927959 and 208, respectively. Further results on the regularity of certain 1-additive sequences will appear in a forthcoming paper.

References

1. Raymond Queneau. "Sur les suites s-additives." *J. Combinatorial Theory Ser. A 12* (1972):31-71; *MR 46*, 1741.
2. Richard K. Guy. *Unsolved Problems in Number Theory*. New York: Springer Verlag, 1981, Problem C4.
3. Marvin C. Wunderlich. "The Improbable Behaviour of Ulam's Summation Sequence." In *Computers and Number Theory*, ed. A. O. L. Atkin & B. J. Birch. New York: Academic Press, 1971, pp. 249-257.
4. Jane A. Hale. *The Lyric Encyclopedia of Raymond Queneau*. Ann Arbor: University of Michigan Press, 1989.

*****