LUCAS NUMBERS AND POLYNOMIALS OF ORDER K AND THE LENGTH OF THE LONGEST CIRCULAR SUCCESS RUN

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1. Introduction

The Lucas numbers L_n, $n = 0, 1, 2, \ldots$, may be defined by

$$L_n = L_{n-1} + L_{n-2},\ n = 2, 3, \ldots,\ L_0 = 2,\ L_1 = 1.$$

Among several combinatorial interpretations of the Lucas numbers in terms of permutations, combinations, compositions (ordered partitions), and distributions of objects into cells, the most commonly used as an alternative combinatorial definition of them is the following: The n^{th} Lucas number L_n, $n = 2, 3, \ldots$, is the number of combinations of n consecutive integers $\{1, 2, 3, \ldots, n\}$ placed on a circle (so that n and 1 are consecutive) with no two integers consecutive. Since

$$L(n, r, 2) = \frac{n}{n-r}\binom{n-r}{r},\ r = 0, 1, 2, \ldots, [n/2],\ n = 2, 3, \ldots,$$

where $[x]$ denotes the integral part of x, is the number of r-combinations of the n consecutive integers $\{1, 2, \ldots, n\}$, placed on a circle, with no two integers consecutive, it is clear that

$$L_n = \sum_{r=0}^{[n/2]} \frac{n}{n-r}\binom{n-r}{r},\ n = 1, 2, \ldots .$$

The polynomials

$$g_n(x) = \sum_{r=0}^{[n/2]} \frac{n}{n-r}\binom{n-r}{r} x^{n-2r},\ n = 1, 2, \ldots,$$

may be called Lucas polynomials. It is worth noting that these polynomials are related to the Chebyshev polynomials,

$$T_n(x) = \cos(n\theta),\ \cos\theta = x,$$

by $g_n(x) = 2i^{-n}T_n(ix/2)$, $i = \sqrt{-1}$. Riordan [8] considered the polynomials $h_n(x) = i^{-n}g_n(ix)$, $n = 1, 2, \ldots$, and the Lucas-type polynomials

$$L_n(x) = \sum_{r=0}^{[n/2]} \frac{n}{n-r}\binom{n-r}{r} x^{n-r} = x^{n/2} g\,(x^{1/2}),\ n = 1, 2, \ldots,$$

in a derivation of Chebyshev-type pairs of inverse relations.

The present paper is motivated by the problem of expressing the distribution function of the length of the longest run of successes in a circular sequence of n independent Bernoulli trials (Philippou & Marki [7]) and the reliability of a circular consecutive k-out-of-n failure system (Derman, Liebermann, & Ross [1]). An elegant solution to this problem is provided by the n^{th} Lucas-type polynomial of order k. This polynomial and the n^{th} Lucas number of order k, as a particular case of it, are examined in Section 2. As probabilistic applications, the above posed problems are discussed in Section 3.

2. Lucas Numbers and Polynomials of order K

Let $L(n, r, k)$ be the number of r-combinations of the n consecutive integers $\{1, 2, \ldots, n\}$ displaced on a circle, with no k integers consecutive. Moser & Abramson [3], essentially showed that

$$(2.1)\qquad L(n, r, k) = \begin{cases} \binom{n}{r}, & r = 0, 1, 2, \ldots, n,\ n = 0, 1, 2, \ldots, k - 1,\ k = 2, 3, \ldots \\ \displaystyle\sum_{j=0}^{[n/k]} (-1)^j \binom{n-r}{j} \frac{n}{n - jk} \binom{n - jk}{n - r}, & r = 0, 1, 2, \ldots, [n - n/k], \\ & n = k, k + 1, \ldots, k = 2, 3, \ldots \\ 0, & r > [n - n/k],\ n = k, k + 1, \ldots, k = 2, 3, \ldots, \end{cases}$$

where $[x]$ denotes the integral part of x. As it can be easily shown, these numbers satisfy the recurrence relation

$$(2.2)\qquad L(n, r, k) = \begin{cases} \displaystyle\sum_{j=1}^{r+1} L(n - j, r - j + 1, k), & r = 0, 1, 2, \ldots, n - 1, \\ & n = 1, 2, \ldots, k,\ k = 2, 3, \ldots \\ \displaystyle\sum_{j=1}^{\min\{r+1, k\}} L(n - j, r - j + 1, k), & r = 0, 1, 2, \ldots, [n - n/k], \\ & n = k + 1, k + 2, \ldots, \\ & k = 2, 3, \ldots . \end{cases}$$

The sum

$$(2.3)\qquad L_{n,k} = \sum_{r=0}^{[n-n/k]} L(n, r, k),\quad n = 1, 2, \ldots, k = 2, 3, \ldots,$$

for $n = k, k + 1, \ldots$, is the number of combinations of the n consecutive integers $\{1, 2, \ldots, n\}$ displaced on a circle, with no k integers consecutive. This number, which for $k = 2$ reduces to $L_{n,2} = L_n$, the n^{th} Lucas number, may be called the n^{th} Lucas number of order k.

The polynomial

$$(2.4)\qquad L_{n,k}(x) = \sum_{r=0}^{[n-n/k]} L(n, r, k)x^{n-r},\quad n = 1, 2, \ldots, k = 2, 3, \ldots$$

may be called the n^{th} Lucas-type polynomial of order k. Clearly,

$$L_{n,k}(1) = L_{n,k}.$$

Recurrence relations, generating functions, and alternative algebraic expressions of these numbers and polynomials and also their connection with the corresponding Fibonacci numbers and polynomials are presented in the following theorems and corollaries.

Theorem 2.1: The sequence $L_{n,k}(x)$, $n = 1, 2, \ldots$, of Lucas-type polynomials of order k satisfies the recurrence relation

$$(2.5)\qquad L_{n,k}(x) = \begin{cases} x\left\{n + \displaystyle\sum_{j=1}^{n-1} L_{n-j,k}(x)\right\}, & n = 2, 3, \ldots, k,\ k = 2, 3, \ldots \\ x\displaystyle\sum_{j=1}^{k} L_{n-j,k}(x), & n = k + 1, k + 2, \ldots, k = 2, 3, \ldots, \end{cases}$$

with $L_{1,k}(x) = x$.

Proof: From (2.4), on using the recurrence relation (2.2), it follows that:

(a) for $n = 1, 2, \ldots, k$,

$$
\begin{aligned}
L_{n,k}(x) &= \sum_{r=0}^{n-1} L(n, r, k)x^{n-r} = \sum_{r=0}^{n-1}\sum_{j=1}^{r+1} L(n-j, r-j+1, k)x^{n-r} \\
&= x\sum_{j=1}^{n}\sum_{r=j-1}^{n-1} L(n-j, r-j+1, k)x^{n-r-1} \\
&= x\left\{n + \sum_{j=1}^{n-1}\sum_{r=j}^{n-1} L(n-j, r-j+1, k)x^{n-r-1}\right\} \\
&= x\left\{n + \sum_{j=1}^{n-1} L_{n-j,k}(x)\right\};
\end{aligned}
$$

(b) for $n = k+1, k+2, \ldots,$

$$
\begin{aligned}
L_{n,k}(x) &= \sum_{r=0}^{[n-n/k]} L(n, r, k)x^{n-r} \\
&= \sum_{r=0}^{[n-n/k]}\ \sum_{j=1}^{\min\{r+1,k\}} L(n-j, r-j+1, k)x^{n-r} \\
&= x\sum_{j=1}^{k}\sum_{r=j-1}^{[n-n/k]} L(n-j, r-j+1, k)x^{n-k-1} \\
&= x\sum_{j=1}^{k} L_{n-j,k}(x);
\end{aligned}
$$

and for $n = 1$,

$$L_{1,k}(x) = L(1, 0, k)x = x.$$

Remark 2.1: The n^{th} Lucas-type polynomial of order k, for $n = 2, 3, \ldots, k$, by virtue of (2.1) and (2.4) may be obtained as

$$L_{n,k}(x) = \sum_{r=0}^{n-1}\binom{n}{r}x^{n-r} = (1+x)^{n-1}. \tag{2.6}$$

Also, from (2.5), for $n = k+1, k+2, \ldots,$ it follows that

$$L_{n,k}(x) = (1+x)L_{n-1,k}(x) - xL_{n-k-1,k}(x). \tag{2.7}$$

Corollary 2.1: The sequence $L_{n,k}$, $n = 1, 2, \ldots,$ of the Lucas numbers of order k satisfies the recurrence relation

$$
L_{n,k} = \begin{cases} n + \sum_{j=1}^{n-1} L_{n-j,k}, & n = 2, 3, \ldots, k,\ k = 2, 3, \ldots \\ \sum_{j=1}^{k} L_{n-j,k}, & n = k+1, k+2, \ldots,\ k+2, 3, \ldots, \end{cases} \tag{2.8}
$$

with $L_{1,k} = 1$.

Theorem 2.2: The generating function of the sequence of Lucas-type polynomials of order k, $L_{n,k}(x)$, $n = 1, 2, \ldots,$ is given by

$$L_k(t; x) = \sum_{n=1}^{\infty} L_{n,k}(x)t^n = \left(x\sum_{j=1}^{k} jt^j\right)\left(1 - x\sum_{j=1}^{k} t^j\right)^{-1}. \tag{2.9}$$

Proof: Multiplying the recurrence relation (2.5) by t^n and summing for $n = 1, 2, \ldots,$ we find

$$\begin{aligned}
L_k(t;\ x) &= \sum_{n=1}^{\infty} L_{n,k}(x)t^n = xt + \sum_{n=2}^{k} L_{n,k}(x)t^n + \sum_{n=k+1}^{\infty} L_{n,k}(x)t^n \\
&= x\sum_{j=1}^{k} jt^j + x\sum_{n=2}^{k}\sum_{j=1}^{n-1} L_{n-j,k}(x)t^n + x\sum_{n=k+1}^{\infty}\sum_{j=1}^{k} L_{n-j,k}(x)t^n \\
&= x\sum_{j=1}^{k} jt^j + x\sum_{j=1}^{k-1}\sum_{n=j+1}^{k} L_{n-j,k}(x)t^n + x\sum_{j=1}^{k}\sum_{n=k+1}^{\infty} L_{n-j,k}(x)t^n \\
&= x\sum_{j=1}^{k} jt^j + x\sum_{j=1}^{k} t^j \sum_{n=j+1}^{\infty} L_{n-j,k}(x)t^{n-j} \\
&= x\sum_{j=1}^{k} jt^j + xL_k(t;\ x)\sum_{j=1}^{k} t^j,
\end{aligned}$$

from which (2.9) follows.

Corollary 2.2: The generating function of the sequence of Lucas numbers of order k, $L_{n,k}$, $n = 1, 2, \ldots$, is given by

$$\text{(2.10)} \quad L_k(t) = \sum_{n=1}^{\infty} L_{n,k}t^n = \left(\sum_{j=1}^{k} jt^j\right)\left(1 - \sum_{j=1}^{k} t^j\right)^{-1}.$$

Theorem 2.3: The n^{th} Lucas-type polynomial of order k may be expressed as

$$\text{(2.11)} \quad (a) \quad L_{n,k}(x) = -1 + \sum_{j=0}^{[n/(k+1)]} (-1)^j \frac{n}{n - jk}\binom{n - jk}{j} x^j (1 + x)^{n-j(k+1)}$$

$$\text{(2.12)} \quad (b) \quad L_{n,k}(x) = \sum \frac{r_1 + 2r_2 + \cdots + kr_k}{r_1 + r_2 + \cdots + r_k} \, \frac{(r_1 + r_2 + \cdots + r_k)!}{r_1!r_2! \ldots r_k!} x^{r_1 + r_2 + \cdots + r_k}$$

where the summation is extended over all partitions of n with no part greater than k, that is over all $r_i = 0, 1, 2, \ldots, n$, $i = 1, 2, \ldots, k$ such that

$$r_1 + 2r_2 + \cdots + kr_k = n.$$

Proof: The generating function (2.9) may be expanded into powers of t as

$$\begin{aligned}
L_k(t;\ x) &= -t\frac{d}{dt}\log\left(1 - x\sum_{j=1}^{k} t^j\right) \\
&= -t\frac{d}{dt}\log\{[1 - (1 + x)t + xt^{k+1}](1 - t)^{-1}\} \\
&= -t(1 - t)^{-1} - t\frac{d}{dt}\log[1 - (1 + x)t + xt^{k+1}] \\
&= -\sum_{n=1}^{\infty} t^n + t\frac{d}{dt}\sum_{r=1}^{\infty}[(1 + x)t - xt^{k+1}]^r/r \\
&= -\sum_{n=1}^{\infty} t^n + t\frac{d}{dt}\sum_{r=1}^{\infty}\sum_{j=0}^{r}(-1)^j\frac{1}{r}\binom{r}{j}x^j(1 + x)^{r-j}t^{r+jk} \\
&= -\sum_{n=1}^{\infty} t^n + \sum_{r=1}^{\infty}\sum_{j=0}^{r}(-1)^j\frac{r + jk}{r}\binom{r}{j}x^j(1 + x)^{r-j}t^{r+jk} \\
&= -\sum_{n=1}^{\infty} t^n + \sum_{n=1}^{\infty}\sum_{j=1}^{[n/(k+1)]}(-1)^j\frac{n}{n - jk}\binom{n - jk}{j}x^j(1 + x)^{n-j(k+1)}t^n
\end{aligned}$$

yielding (2.11).

A different expansion of (2.9) as

$$L_k(t;\, x) = -t\frac{d}{dt}\log\left(1 - x\sum_{j=1}^{k} t^j\right) = t\frac{d}{dt}\sum_{n=1}^{\infty}\left(x\sum_{j=1}^{k} t^j\right)^r\Big/ r$$

$$= t\frac{d}{dt}\sum_{r=1}^{\infty}\sum\frac{(r-1)!}{r_1!r_2!\,\ldots\,r_k!}\,x^{r_1+r_2+\cdots+r_k}\,t^{r_1+2r_2+\cdots+kr_k}$$

$$= \sum_{r=1}^{\infty}\sum\frac{(r_1+2r_2+\cdots+kr_k)(r-1)!}{r_1!r_2!\,\ldots\,r_k!}\,x^{r_1+r_2+\cdots+r_k}\,t^{r_1+2r_2+\cdots+kr_k}$$

where in the inner sums the summation is extended over all $r = 0, 1, 2, \ldots, r$, $i = 1, 2, \ldots, k$, such that $r_1 + r_2 + \cdots + r_k = r$, on putting

$$n = r - \sum_{j=1}^{k}(j-1)r_j$$

yields

$$L_k(t;\, x) = \sum_{n=1}^{\infty}\left\{\sum\frac{r_1+2r_2+\cdots+kr_k}{r_1+r_2+\cdots+r_k}\;\frac{(r_1+r_2+\cdots+r_k)!}{r_1!r_2!\,\ldots\,r_k!}\,x^{r_1+r_2+\cdots+r_k}\right\}t^n$$

where in the inner sum the summation is extended over all $r_i = 0, 1, 2, \ldots, n$, $i = 1, 2, \ldots, k$, such that $r_1 + 2r_2 + \cdots + kr_k = n$. The last expression implies (2.12).

Corollary 2.3: The n^{th} Lucas number of order k may be expressed as

$$(2.13)\quad (a)\quad L_{n,k} = -1 + \sum_{j=0}^{[n/(k+1)]}(-1)\,\frac{n}{n-jk}\binom{n-jk}{j}2^{n-j(k+1)},$$

$$(2.14)\quad (b)\quad L_{n,k} = \sum\frac{r_1+2r_2+\cdots+kr_k}{r_1+r_2+\cdots+r_k}\;\frac{(r_1+r_2+\cdots+r_k)!}{r_1!r_2!\,\ldots\,r_k!}$$

where the summation is extended over all $r_i = 0, 1, 2, \ldots, n$ such that

$$r_1 + 2r_2 + \cdots + kr_k = n.$$

Remark 2.2: A known expression for the n^{th} Lucas number L_n and two expressions for the n^{th} Lucas number of order 3, $H_n \equiv L_{n,3}$, may be deduced from the general expression (2.14). Setting $k = 2$ and introducing the variable $r = r_2$, it follows that

$$L_n = \sum_{r=0}^{[n/2]}\frac{n}{n-r}\binom{n-r}{r}.$$

Putting $k = 3$ and introducing the variables $r = r_2$, $j = r_3$, (2.14) reduces to

$$(2.15)\quad H_n = \sum_{r=0}^{[n/2]}\sum_{j=0}^{[(n-2r)/3]}\frac{n}{n-r-2j}\binom{n-r-2j}{r+j}\binom{r+j}{r}$$

while, introducing the variables $r = r_2 + 2r_3$, $j = r_3$, (2.20) becomes

$$(2.16)\quad H_n = \sum_{r=0}^{[2n/3]}\sum_{j=0}^{[r/3]}\frac{n}{n-r}\binom{n-r}{j}\binom{n-r-j}{r-2j}.$$

The Lucas numbers L_n are related to Fibonacci numbers F_n by

$$L_n = F_n + 2F_{n-1} = F_{n+1} + F_{n-1}.$$

An extension of this relation to the Lucas-type polynomials and the Fibonacci-type polynomials (see [6]) is obtained in the following theorem.

Theorem 2.4: The Lucas-type polynomials of order k, $L_{n,k}(x)$, $n = 1, 2, \ldots$, are expressed in terms of the Fibonacci-type polynomials of order k, $F_{n,k}(x)$, $n = 1, 2, \ldots$, by

$$(2.17)\quad L_{n,k}(x) = x \sum_{j=1}^{\min\{n,k\}} jF_{n-j+1,k}(x),\ n = 1, 2, \ldots,\ k = 2, 3, \ldots .$$

Proof: Since (see [6])

$$\sum_{n=0}^{\infty} F_{n+1,k}(x)t^n = \left(1 - x\sum_{j=1}^{k} t^j\right)^{-1},$$

it follows from (2.9) that

$$\begin{aligned}\sum_{n=1}^{\infty} L_{n,k}(x)t^n &= x\left(\sum_{j=1}^{k} jt^j\right)\left(\sum_{r=0}^{\infty} F_{r+1,k}(x)t^r\right)\\ &= \sum_{n=1}^{\infty}\left\{x\sum_{j=1}^{\min\{n,k\}} jF_{n-j+1,k}(x)\right\}t^n,\end{aligned}$$

which implies (2.17).

Corollary 2.4: The Lucas numbers of order k are expressed in terms of the Fibonacci numbers of order k by

$$(2.18)\quad L_{n,k} = \sum_{j=1}^{\min\{n,k\}} jF_{n-j+1,k},\ n = 1, 2, \ldots,\ k = 2, 3, \ldots .$$

Remark 2.3: The polynomial

$$(2.19)\quad g_{n,k}(x) = \sum_{r=0}^{[n-n/k]} L(n, r, k)x^{(n-r)k-n},\ n = 1, 2, \ldots,\ k = 2, 3, \ldots,$$

may be called the n^{th} Lucas polynomial of order k. It is related to the Lucas-type polynomial (2.4) by

$$(2.20)\quad g_{n,k}(x) = x^{-n}L_{n,k}(x^k),\ n = 1, 2, \ldots,\ k = 2, 3, \ldots .$$

Expressions for these polynomials, analogous to (2.5), (2.9), (2.11) and (2.12), on using (2.20), may easily be deduced. Further,

$$(2.21)\quad g_{n,k}(x) = \sum_{j=1}^{\min\{n,k\}} jx^{k-j+1}f_{n-j+1,k}(x),\ n = 1, 2, \ldots,\ k = 2, 3, \ldots,$$

where $f_{n,k}(x)$ is the n^{th} Fibonacci polynomial of order k (see [5] and [2] as k-bonacci polynomial). This relation may be deduced from (2.17) by virtue of (2.20) and [4],

$$f_{n,k}(x) = x^{-n+1}F_{n,k}(x^k).$$

3. Probabilistic Applications

Consider a circular sequence of n independent Bernoulli trials with constant success probability p and let $q = 1 - p$. Further, let C_n be the length of the longest circular run of successes and let S_n be the total number of successes. In Theorem 3.1, the conditional distribution function of C_n, given $S_n = r$, $P(C_n \le x/S_n = r)$, $-\infty < x < \infty$, is obtained in terms of the numbers $L(n, r, [x] + 1)$ and the distribution function of C_n, $P(C_n \le x)$, $-\infty < x < \infty$, is expressed in terms of the Lucas-type polynomials of order $[x] + 1$.

Theorem 3.1: Let C_n and S_n be the length of the longest run of successes and total number of successes, respectively, in a circular sequence of n independent Bernoulli trials with constant success probability p. Then,

$$(3.1)\qquad P(C_n \le x/S_n = r) = \begin{cases} 0 & \\ L(n, r, k+1)/\binom{n}{r}, & 0 \le x < r \le n,\ k = [x] \\ 1, & r \le x < \infty,\ r \le n. \end{cases}$$

$$(3.2)\qquad P(C_n \le x) = \begin{cases} 0, & -\infty < x < 0 \\ p^n L_{n,k+1}(q/p), & 0 \le x < n \\ 1, & n \le x < \infty. \end{cases}$$

Proof: The elements of the sample space are combinations $\{i_1, i_2, \ldots\}$ of the n consecutive integers $\{1, 2, \ldots, n\}$ displaced on a circle where i_m is the position of the m^{th} success, $m = 1, 2, \ldots$. The event $\{C_n \le x, S_n = r\}$ contains all the r-combinations of the n integers $\{1, 2, \ldots, n\}$ displaced on a circle, with no $k + 1 = [x] + 1$ integers consecutive. Clearly, the number of these r-combinations is given by $L(n, r, k + 1)$. Further, each of these r-combinations has probability $p^r q^{n-r}$. Hence,

$$(3.3)\qquad P(C_n \le x,\ S_n = r) = L(n, r, k+1)p^r q^{n-r},\ k = [x],$$

and since

$$P(S_n = r) = \binom{n}{r}p^r q^{n-r},\ r = 0, 1, 2, \ldots, n,$$

(3.1) follows.

Summing the probabilities (3.3) for $r = 0, 1, 2, \ldots, [n - n/(k + 1)]$, on using (2.4), (3.2) is deduced.

Since $P(C_n = k) = P(C_n \le k) - P(C_n \le k - 1)$, $k = 0, 1, 2, \ldots$, on using (3.1), the next corollary is deduced.

Corollary 3.1: The probability function of the random variable C is given by

$$(3.4)\qquad P(C_n = k) = \begin{cases} q^n, & k = 0 \\ p^n, & k = n \\ p^n\{L_{n,k+1}(q/p) - L_{n,k}(q/p)\}, & k = 1, 2, \ldots, n-1. \end{cases}$$

Remark 3.1: A circular consecutive-k-out-of-n: F system is a system of n components displaced on a circle which fails when k consecutive components fail. Suppose that the probability for each component to function is p and to fail is $q = 1 - p$. Derman, Lieberman, and Ross (see [1]) expressed its reliability $R_c(p, n, k)$ as

$$R_c(p, k, n) = p^2 \sum_{j=1}^{k} j q^{j-1} R_L(p, k, n - j - 1),$$

where $R_L(p, k, n)$ denotes the reliability of a linear consecutive-k-out-of-n: F system.

Interpreting as a "success" the failure of a component, the reliability $R_c(p, k, n)$ is the probability that the length C_n of the longest circular run of successes in a circular sequence of n independent Bernoulli trials with constant success probability 1 is less than or equal to k. It is then clear from Theorem 3.1 that

$$(3.5)\qquad R_c(p, k, n) = q^n L_{n,k}(p/q) = \sum_{j=0}^{[n/(k+1)]} (-1)^j \frac{n}{n - jk}\binom{n - jk}{j} p^j q^{jk} - q^n$$

with the last equality by (2.11).

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