PERIODIC FIBONACCI AND LUCAS SEQUENCES

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1. Introduction

In the early thirteenth century there appeared the book *Liber Abaci* by the mathematician Leonardo of Pisa [7], who also became known as Fibonacci (see also [2]). In it a problem concerning an ideal case of the reproduction of rabbits is treated, and the sequence

(1) $F = 1, 2, 3, 5, 8, \ldots$

is introduced. This sequence has since become known as the *Fibonacci Sequence*. One of its features is the recurrence relation

(2) $a_n = a_{n-1} + a_{n-2}$, for $n \geq 3$.

In the second half of the nineteenth century E. Lucas [8], who had actually coined the term *Fibonacci Numbers*, introduced a similar sequence connected closely to that of Fibonacci,

(3) $L = 1, 3, 4, 7, 11, \ldots,$

obeying the same recurrence relation as F. The sequence L has since become known as the *Lucas Sequence* [3] (see also [4]).

Since then the *generalized* sequences of both kinds have been introduced. For both, the recurrence relation is

$$a_n = aa_{n-1} + \sigma a_{n-2},$$

where a and σ are prescribed numbers.

We shall also stipulate $a_0 = 1$ or 2 according to whether the sequence is a generalized F or a generalized L, respectively. The recurrence relation holds already for $n = 2$ (see also [3]). In [10] Wall treated generalized Fibonacci sequences modulo an integer m and showed that some are periodic mod (m) (see also [6], [11], and [12]).

Now let a and σ be two arbitrary complex numbers and let the terms of the generalized Fibonacci (Lucas) sequence be $f_0 = 1$, $f_1 = a$ ($g_0 = 2$, $g_1 = a$). It turns out that in some cases such sequences are periodic. Put, for example, $a = 1$, $\sigma = -1$. Then both sequences are periodic of period 6.

In this paper we wish to characterize those sequences which are periodic; in other words, to specify precisely for which ordered pair (a, σ) the corresponding Fibonacci (Lucas) sequence is periodic. We shall also specify in each relevant case the period T, T being the least positive integer for which $a_{n+T} = a_n$ for every n.

Let us first look at degenerate cases. The case $a = \sigma = 0$ is trivial with $T = 0$. If just one of the two vanishes, the remaining parameter is necessarily a root of unity, a trivial case being $a = 1$, $\sigma = 0$, $T = 1$.

We may, therefore, assume both parameters to be nonzero.

2. Periodic Row-Column Matrices

Let $n > 1$ be a positive integer. Consider an $n \times n$-matrix $A = (a_{ij})$ over the complex field with $a_{ij} = 0$ if both i and j are greater than one. Put

$$a_{11} = a, \quad \sum_{j=2}^{n} a_{1j}a_{j1} = \sigma.$$

We shall name such a matrix a (one-row)-(one-column) matrix or, in short, an RCM.

The characteristic polynomial of A is $\lambda^n - a\lambda^{n-1} + \sigma\lambda^{n-2}$ so that the two nonzero eigenvalues of A satisfy the quadratic equation

(4) $\quad \lambda^2 - a\lambda - \sigma = 0$

whose roots are

$$\lambda_{1,2} = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 + \sigma}.$$

It follows that for $n \geq 2$ the spectrum of A depends solely on a and σ and is independent of n.

For $\sigma = a^2/4$, the matrix A is neither diagonalizable nor periodic for any nonzero value of a.

The polynomial $f(z) = z^2 - az - \sigma$ appears in a paper by M. Ward [11], among others. Ward also considers what he calls *degenerate* sequences in which zeros appear periodically, with periods 2, 3, 4, and 6, although the sequences as such are not periodic (see, e.g., [11, Th. 3]).

Except for the case $\sigma = -a^2/4$, the two nonvanishing eigenvalues of A are distinct. In addition, we have rank $A = 2$, and hence, A is diagonalizable. For $i = 1, 2$, we have

(5) $\quad \lambda_i^2 = a\lambda_i + \sigma,$

(6) $\quad \lambda_1 + \lambda_2 = a.$

Let j be a positive integer. Define

$$\gamma_j = \text{Tr } A^j.$$

We have

$$\gamma_1 = a.$$

$$\gamma_2 = \lambda_1^2 + \lambda_2^2 = a\lambda_1 + \sigma + a\lambda_2 + \sigma = a^2 + 2\sigma.$$

Also, for $j \geq 3$, equalities (1) and (2) imply

(7) $$\gamma_j = \lambda_1^j + \lambda_2^j = \lambda_1^{j-2}\lambda_1^2 + \lambda_2^{j-2}\lambda_2^2 = a\lambda_1^{j-1} + \sigma\lambda_1^{j-2} + a\lambda_2^{j-1} + \sigma\lambda_2^{j-2}$$
$$= a\gamma_{j-1} + \sigma\gamma_{j-2}.$$

We thus have a recurrence formula for γ_j, $j \geq 3$, displaying a generalized Fibonacci sequence. We now turn to the possible periodicity of an RCM. A necessary condition for A to be periodic is $|\lambda_1| = |\lambda_2| = 1$. It also follows that A is periodic if and only if γ_k is periodic.

Putting

$$\sqrt{\frac{a^2 + \sigma}{4}} = w,$$

we have

$$\lambda_1 = \frac{a}{2} + w, \quad \lambda_2 = \frac{a}{2} - w.$$

For both λ_1 and λ_2 to be on the unit circle, it is necessary that

$$|w| = \sqrt{1 - \frac{|a|^2}{4}} \quad \text{and} \quad \arg w = \arg a \pm \frac{\pi}{2}.$$

Set $\arg a = \phi$ and $\arg \lambda_1 - \phi = \psi$. Then $\arg \lambda_2 = \arg \lambda_1 - 2\psi$, so that

$$\arg \lambda_1 = a + \psi \quad \text{and} \quad \arg \lambda_2 = a = \psi \quad \text{(see Fig. 1).}$$

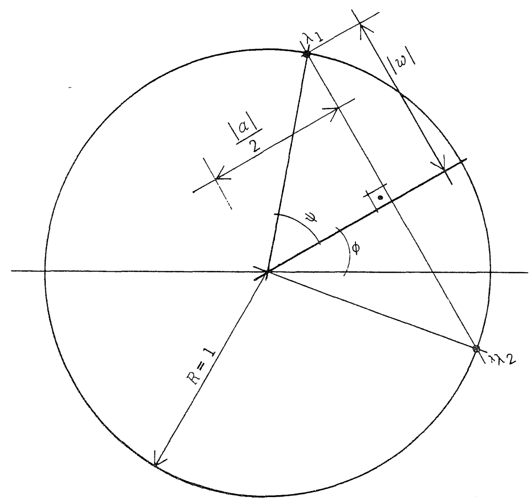

FIGURE 1

Then

$$\tan\psi = \frac{\sqrt{1 - \frac{|a|^2}{4}}}{\frac{|a|}{2}} = \sqrt{\frac{4}{|a|^2} - 1}.$$

Now set

$$\pm\psi + \phi = \text{arc tan}\left(\pm\sqrt{\frac{4}{|a|^2} - 1}\right) + \arg a = \frac{2\pi}{\rho_i}, \tag{8}$$

where $i = 1$ for the plus sign and $i = 2$ for the minus sign. A necessary and sufficient condition for A to be periodic is that both λ_1 and λ_2 be roots of unity. We also find that equation (4) implies

$$\sigma = \lambda^2 - a\lambda = \lambda(\lambda - a) = \frac{a}{2} \pm \left(i\sqrt{1 - \frac{|a|^2}{4}}e^{i\phi}\right)\left(-\frac{a}{2} \pm i\sqrt{1 - \frac{|a|^2}{4}}e^{i\phi}\right)$$

$$= \frac{a^2}{4} - \left(1 - \frac{|a|^2}{4}\right)e^{2i\phi} = \frac{|a|^2}{4}e^{2i\phi} - \frac{a^2}{4} - e^{2i\phi} = -e^{2i\phi}.$$

We thus have

Theorem 1: Let A be an RCM. Then A is periodic if and only if

(i) for both choices ($\pm$) we have $\pi^{-1}\left(\arg a \pm \text{arc tan}\sqrt{\frac{4}{|a|^2} - 1}\right)$ are rational;

(ii) $\sigma = -e^{2i \arg a}$.

Corollary 1: Let A be an RCM. Then A is periodic if and only if the following three conditions hold.

(i) $\pi^{-1} \arg a$ is rational;

(ii) π^{-1} arc tan $\sqrt{\frac{4}{|a|^2} - 1}$ is rational;

(iii) $\sigma = -e^{2i \arg a}$.

Corollary 2: Let A be a real RCM. Then A is periodic if and only if

$$\pi^{-1} \text{ arc tan}\sqrt{\frac{4}{a^2} - 1} \text{ is rational and } \sigma = -1.$$

Corollary 3: A real RCM is periodic if and only if

$$\pi^{-1} \text{ arc } \tan\sqrt{\frac{4}{a^2} - 1} \quad \text{and} \quad \sigma = -1.$$

Corollary 4: Let A be a purely imaginary RCM. Then A is periodic if and only if

$$\pi^{-1} \text{ arc } \tan\sqrt{-\frac{4}{a^2} - 1} \text{ is rational} \quad \text{and} \quad \sigma = 1.$$

Corollary 5: A necessary condition for an RCM to be periodic is that a satisfy the inequality $0 < |a| < 2$.

Corollary 6: A necessary condition for an RCM to be periodic is $|\sigma| = 1$.

Let us now seek the period $T = T(A)$. It will clearly be the least integral for which both $T(\phi + \psi)$ and $T(\phi - \psi)$ are integral multiples of 2π. Put

$$\phi + \psi = \frac{2\pi}{\rho_1}, \quad \phi - \psi = \frac{2\pi}{\rho_2}.$$

For $i = 1, 2$, the ρ_i are necessarily rational, so that we may put

$$\rho_i = \frac{m_i}{n_i}, \text{ with } (m_i, n_i) = 1.$$

We then have

Theorem 2: Let A be a given periodic RCM. Then the period $T(A)$ is given by the formulas $T(A) = \text{L.C.M.}(m_1, m_2)$ where the m_i are defined as above.

We also have, for a periodic RCM, $(|a|/2) = \cos\psi$, so that we may write

$$a = 2\cos\psi\, e^{i\phi}. \tag{9}$$

We may also write $\lambda_1 = e^{i(\phi+\psi)}$, $\lambda_2 = e^{i(\phi-\psi)}$, so that

$$\lambda_1 + \lambda_2 = e^{i\phi}(e^{i\psi} + e^{-i\psi}) = 2\cos\psi\, e^{i\phi}.$$

Then it is easy to see that $\lambda_1^k = e^{ki(\phi+\psi)}$, $\lambda_2^k = e^{ki(\phi-\psi)}$ so that, likewise,

$$\gamma_k = \lambda_1^k + \lambda_2^k = 2\cos(k\psi)e^{ki\phi},$$

thus proving that A is periodic if and only if the traces of the powers of A are periodic. We then have

Corollary 7: Let A be a periodic RCM with $a = 1$. Then A has period 6.

Proof: We have $\phi = 0$ and $\cos\psi = 1/2$, so that $\psi = \pi/3$. The result follows.

Let us consider two examples.

Example 1: Let $\phi = \frac{\pi}{20}$, $\psi = \frac{13}{60}\pi$. Then

$$a = 2\cos\frac{13}{60}\pi\, e^{\frac{\pi i}{20}}, \quad \sigma = -e^{\frac{\pi i}{10}}.$$

We also have $\phi + \psi = \frac{4}{15}\pi$, $\phi - \psi = -\pi/6$, so that $m_1 = 15$, $m_2 = 12$, and hence,

$$T = \text{L.C.M.}(15, 12) = 60.$$

Example 2: Let $a = e^{\pi i/3}$. Then $= -e^{2\pi i/3}$. Also $\cos\psi = 1/2$ so that $\phi = \psi = \pi/3$; hence, $\phi + \psi = 2\pi/3$, $\phi - \psi = 2\pi$, $m_1 = 3$, $m_2 = 1$, and so $T = 3$.

3. The Leading Element of a Power of an RCM

Let A be an RCM. Put $A = (a_{ij})$. Let $a_{ij}^{(k)}$ denote the (i, j)-element of A^k. We consider $a_{11}^{(k)}$ for $k > 1$. Put $a_{ij} = \alpha_j$, $a_{i1} = \beta_i$. We then have $a_{11}^{(2)} = a^2 + \sigma$.

For $i \neq 1 \neq j$, we have

$$a_{1j}^{(2)} = a\alpha_j,\ a_{i1}^{(2)} = a\beta_i,\ a_{ij}^{(2)} = \beta_i\alpha_j,$$
$$a_{11}^{(3)} = a^3 + 2a\sigma,\ a_{1j}^{(3)} = (a^2 + \sigma)\alpha_j$$
$$a_{i1}^{(3)} = (a^2 + \sigma)\beta_i,\ a_{ij}^{(3)} = a\beta_i\alpha_j.$$

Put $f_0 = 1$, $f_1 = a$, $f_2 = a^2 + \sigma$.

Suppose that for some k we have

$$a_{11}^{(k)} = f_k,\ a_{1j}^{(k)} = \alpha_j f_{k-1}, \tag{10}$$
$$a_{i1}^{(k)} = \beta_i f_{k-1},\ a_{ij}^{(k)} = \beta_i\alpha_j f_{k-2} \quad \text{for } i \neq 1 \neq j.$$

Then

$$a_{11}^{(k+1)} = af_k + \sigma f_{k-1} = f_{k+1},$$
$$a_{1}^{(k+1)} = \alpha_j(af_{k-1} + \sigma f_{k-2}) = \alpha_j f_k,$$
$$a_{i1}^{(k+1)} = \beta_i f_k,$$
$$a_{ij}^{(k+1)} = \beta_i\alpha_j f_{k-1}.$$

We may use induction since 10 holds for $k = 2$. We thus have

Lemma 1: Let A be an RCM. Then equalities (10) hold for every $i, j > 1$ and for $k \geq 2$.

We thus obtain

Theorem 3: Let A be an RCM. Then the leading elements and the traces of the successive powers of A form a generalized Fibonacci sequence and a generalized Lucas sequence.

For $a = \sigma = 1$ we obtain the original Fibonacci and Lucas sequences appearing in (1) and (2). We may therefore look at RCM's as generating Fibonacci and Lucas sequences. A particular such case has already been treated in [5] and also in [1].

We may now combine the two aspects of RCM's, namely, periodicity on the one hand, and Fibonacci sequences on the other in order to draw the following conclusion.

Theorem 4: A generalized Fibonacci (Lucas) sequence with complex parameters a and σ is periodic if and only if both

$$\pi^{-1} \text{ arc tan} \sqrt{\frac{4}{|a|^2} - 1} \quad \text{and} \quad \pi^{-1} \arg a$$

are rational and $\sigma = -e^{2i \arg a}$.

Corollary 8: A generalized Fibonacci (Lucas) sequence with real parameter a is periodic if and only if

$$\pi^{-1} \text{ arc tan} \sqrt{\frac{4}{a^2} - 1}$$

is rational and $\sigma = -1$. The period T is determined as prescribed by Theorem 2.

Let $n \geq 2$ be an integer. Consider a generalized Fibonacci or Lucas sequence for which the parameters ϕ and ψ are $\phi = \psi = \pi/n$. Then

$$\phi + \psi = \frac{2\pi}{n},\ \phi - \psi = 2\pi$$

so that

$$a = 2 \cos\frac{\pi}{n} e^{\frac{\pi i}{n}},\ \sigma = -e^{\frac{-2\pi i}{n}};$$

so we get a periodic sequence of period n. We may thus state

Corollary 9: Every positive integer ≥ 2 is a period for some generalized Fibonacci (Lucas) sequence.

For $n = 2$, we have to stipulate $a = 0$, $\sigma = 1$, since $\phi = \psi = \pi/2$. We may also state

Corollary 10: Every positive integer is a period for some RCM.

For $n = 1$ choose $a = 1$, $\sigma = 0$. The generalized Fibonacci sequence with parameters a and σ suggest that the traces γ_k be polynomials in a, σ of degree k, so that

$$\gamma_k = \sum_{j=0}^{\lfloor k/2 \rfloor} \phi_{kj} a^{k-2j} \sigma^j.$$

The coefficients ϕ_{kj} may be established by graph-theoretical counting techniques. Induction may also be used to show that

$$\phi_{kj} = \binom{k-j}{j} + \binom{k-j-1}{j-1} = k\frac{(k-j-1)!}{j!(k-2j)!}.$$

The verification is left to the reader.

A similar formula may be found in [9].

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