MORE ON FIBONACCI NIM

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Fibonacci Nim [1] was originally stated as follows:

> "Consider a game involving two players in which initially there is a group of 100 or less objects. The first player may reduce the pile by any Fibonacci number (member of the series 1, 1, 2, 3, 5, 8, 13, 21, ...). The second player does likewise. The player who makes the last move wins the game."

Let persons A and B be playing the game which A wins. If A is to win he must be able to reduce the pile to zero on his final move. Thus A must draw from $0+F_n$ ($n = 1, 2, 3, \ldots$) on his final move.

Looking at the sequence of the number of objects from which A must draw to win on the final move, 1, 2, 3, 5, ..., we see that 4 is the first positive integer missing. If B is forced to play with 4 objects remaining, A can certainly win the game.

Now suppose A gets the opportunity to draw from $4+F_n$(5, 6, 7, 9, 12, ...). A will be able to reduce the pile to 4 objects and can continue to win.

The smallest positive integer that is not contained in the union of the sets $\{0+F_n\}$ and $\{4+F_n\}$ is 10. If B is forced to draw from a pile of 10 objects, B cannot reduce the pile to 4 or 0 but B will leave A in a position to reduce the pile to 4 or 0 and thus A can win.

Now we wish to generate the sequence of positions from which it is unsafe to draw (0, 4, 10, ...). Let $U_1=0$. Then U_2 is the smallest positive integer which is not equal to U_1+F_n ($n = 2, 3, \ldots$). U_3 is the smallest positive integer which is not equal to U_1+F_n or U_2+F_n ($n = 2, 3, \ldots$).

Therefore U_r ($r = 2, 3, \ldots$) is the smallest positive integer which is not equal to U_t+F_n, where $t = 1, 2, \ldots, r-1$ and $n = 2, 3, \ldots$

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U_r

F_n	0	1	2	3	5	8	13	21	34	55	89
	4	5	6	7	9	12	17	25	38	59	93
	10	11	12	13	15	18	23	31	44	65	99
	14	15	16	17	19	22	27	35	48	69	
	20	21	22	23	25	28	33	41	54	75	
	24	25	26	27	29	32	37	45	58	79	
	30	31	32	33	35	38	43	51	64	85	
	36	37	38	39	41	44	49	57	70	91	
	40	41	42	43	45	48	53	61	74	95	
	46	47	48	49	51	54	59	67	80		
	50	51	52	53	55	58	63	71	84		
	56	57	58	59	61	64	69	77	90		
	60	61	62	63	65	68	73	81	94		
	66	67	68	69	71	74	79	87	100		
	72	73	74	75	77	80	85	93			
	76	77	78	79	81	84	89	97			
	82	83	84	85	87	90	95				
	86	87	88	89	91	94	99				
	92	93	94	95	97	100					
	96	97	98	99							

The first player can always win if he starts on some position not equal to U_r ($r = 1, 2, \ldots$) and always reduces the pile to some U_r.

Here are all the values of U_r thus far computed:

0	4	10	14	20	24	30	36	40	46	50	56	60	66	72
76	82	86	92	96	102	108	112	118	122	128	132	138	150	160
169	176	186	192	196	202	206	212	218	222	228	232	238	242	248
254	260	264	270	274	280	284	290	296	300	306	310	316	322	326
332	338	342	348	352	358	364	368	374	378	384	388	394	400	406
410	416	420	426	430	436	442	446	452	456	462	468	472	478	484
488	494	498	504	510	514	520	524	530	534	540	552	556	562	566
572	576													

The following observations can be made:

1. $U_{r+1} = U_r +$ some non-Fibonacci number.
2. If $U_{r+1} - U_r = 4$, then $U_{r+2} - U_{r+1} \neq 4$ since $4+4=8=F_6$.
3. Thus the average difference of $U_{r+1} - U_r \geqq 5$, $r = 1, 2, 3, \ldots$
4. The density of $\{U_r\}$ in the positive integers must be $\leqq 1/5$.
5. The probability that the starting person can win is $\geqq 4/5$ if nothing is known about the starting position of the game.

The following questions are left unanswered:

1. Is there a closed form solution for $\{U_r\}$?

2. What is the limiting density of $\{U_r\}$ in the positive integers?

Similar results are found when one considers "Lucas Nim" analogous to Fibonacci Nim.

REFERENCES

1. Brother U. Alfred, "Research Project: Fibonacci Nim," Fibonacci Quarterly, 1(1963), No. 1, p. 63.
2. Michael J. Whinihan, "Fibonacci Nim," Fibonacci Quarterly, 1(1963), No. 4, pp. 9-13.

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ON SUMS $F_x^2 \pm F_y^2$

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Formulas for the sum of the squares of Fibonacci numbers are:

(1) $$F^2_{n+2k} + F^2_n = F_{n+2k-2}\, F_{n+2k+1} + F_{2k-1}\, F_{2n+2k-1}$$

(2) $$F^2_{n+2k+1} + F^2_n = F_{2k+1}\, F_{2n+2k+1}$$

(3) $$F^2_{n+2k} - F^2_n = F_{2k}\, F_{2n+2k}$$

(4) $$F^2_{n+2k+1} - F^2_n = F_{n-1}\, F_{n+2} + F_{2k}\, F_{2n+2k+2}$$

Validity of the above is established by using:

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n),\ L_n = \alpha^n + \beta^n,\ \alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2},\ \alpha\beta = -1\ .$$

For example:

$$5(F^2_{n+2k+1} - F^2_n) =$$
$$(\alpha^{2n+4k+2} + \beta^{2n+4k+2}) - (\alpha^{2n} + \beta^{2n}) - 2\alpha^n\beta^n(\alpha^{2k+1}\beta^{2k+1} - 1) =$$
$$L_{2n+4k+2} - L_{2n} - 2(-1)^n(-2) = L_{2n+4k+2} - L_{2n} - (-1)^{n-1}L_3$$

$$5(F_{n-1}\, F_{n+2} + F_{2k}\, F_{2n+2k+2}) =$$
$$(\alpha^{2n+1} + \beta^{2n+1}) + (\alpha^{2n+4k+2} + \beta^{2n+4k+2}) - \alpha^{n-1}\beta^{n-1}(\alpha^3+\beta^3) -$$
$$- \alpha^{2k}\beta^{2k}(\alpha^{2n+2}+\beta^{2n+2}) =$$
$$L_{2n+4k+2} + (L_{2n+1} - L_{2n+2}) - (-1)^{n-1}L_3 = L_{2n+4k+2} - L_{2n} - (-1)^{n-1}L_3.$$

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