ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-59 *Proposed by D.W. Robinson, Brigham Young University, Provo, Utah*

Show that, if $m > 2$, then the period of the Fibonacci sequence $0, 1, 1, 2, 3, \ldots, F_n, \ldots$ reduced modulo m is twice the least positive integer n such that $F_{n+1} \equiv (-1)^n F_{n-1} \pmod{m}$.

H-60 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

It is well known that if p_k is the least integer such that $F_{n+p_k} \equiv F_n \bmod 10^k$, then $p_1 = 60$, $p_2 = 300$ and $p_k = 1.5 \times 10^k$ for $k \geq 3$. If $Q(n, k)$ is the kth digit of the nth Fibonacci, then for fixed k, $Q(n, k)$ is periodic, that is q_k is the least integer such that $Q(n+q_k, k) \equiv Q(n, k) \bmod 10$. Find an explicit expression for q_k.

H-61 *Proposed by P.F. Byrd, San Jose State College, San Jose, California*

Let $f_{n,k} = 0$ for $0 \leq n \leq k-2$, $f_{k-1,k} = 1$ and

$$f_{n,k} = \sum_{j=1}^{k} f_{n-j,k} \quad \text{for} \quad n \geq k \; .$$

Show that

$$\frac{1}{2} < \frac{f_{n,k}}{f_{n+1,k}} < \frac{1}{2} + \frac{1}{2k} \quad \text{for} \quad n \geq 1 \; .$$

Hence

$$\lim_{k \to \infty} \; \lim_{n \to \infty} \; \frac{f_{n,k}}{f_{n+1,k}} = \frac{1}{2}$$

See E. P. Miles, "Generalized Fibonacci Numbers and their Associated Matrices," The American Mathematical Monthly, Vol. 67, No. 8.

H-62 *Proposed by H.W. Gould, West Virginia University, Morgantown, West Va.*

Find all polynomials $f(x)$ and $g(x)$, of the form

$$f(x+1) = \sum_{j=o}^{r} a_j x^j, \quad a_j \text{ an integer}$$

$$g(x) = \sum_{j=o}^{s} b_j x^j, \quad b_j \text{ an integer}$$

such that

$$2\left\{x^2 f^3(x+1) - (x+1)^2 g^3(x)\right\} + 3\left\{x^2 f^2(x+1) - (x+1)^2 g^2(x)\right\}$$
$$+ 2(x+1)\left\{xf(x+1) - (x+1)g(x)\right\} = 0 .$$

H-63 *Proposed by Stephen Jerbic, San Jose State College, San Jose, California*

Let

$$F(m, o) = 1 \text{ and } F(m, n) = \frac{F_m F_{m-1} \cdots F_{m-n+1}}{F_n F_{n-1} \cdots F_1} \quad o < n \leq m ,$$

be the Fibonomial coefficients, where F_n is the nth Fibonacci number. Show

$$\sum_{n=o}^{2m-1} F(2m-1, n) = \prod_{i=o}^{m-1} L_{2i}, \quad m \geq 1 .$$

H-64 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Show

$$F_{n+1} = \prod_{j=1}^{n} \left(1 - 2i \cos \frac{j\pi}{n+1}\right) ,$$

where F_n is the nth Fibonacci number.

ALL THE SOLUTIONS

H-30 *Proposed by J. A. H. Hunter, Toronto, Ontario, Canada*

Find all non-zero integral solutions to the two Diophantine equations,

(a) $$x^2 + xy + x - y^2 = 0$$

(b) $$x^2 - xy - x - y^2 = 0 \quad .$$

Solution by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

We first observe that (x_o, y_o) is a solution of (a), if and only $(-x_o, y_o)$ is a solution of (b). Thus we may limit our considerations to just one of the equations, say (b).

Equation (b) has the form

$$x^2 - (y+1)x - y^2 = 0$$

which, considering y as a parameter, has solutions

$$x = \frac{(y+1) \pm \sqrt{(y+1)^2 + 4y^2}}{2}$$

For x to be an integer, it is clearly necessary and sufficient that $(y+1)^2 + 4y^2$ be a perfect square, that is, there exists an integer z such that

$$(y+1)^2 + 4y^2 = z^2 \quad ,$$

or,

$$\boxed{(y+1)^2 + (2y)^2 = z^2 \ .}$$

Let us look first for solutions with $y > 0$. Note that $2y/d$ and $(y+1)/d$ are relatively prime integers, where $d \geq 1$ is the greatest common divisor of $2y$ and $y+1$, so that, by the well-known theorem on solutions of $x^2 + y^2 = z^2$, there exist two relatively prime positive integers r and s of different parity, with $r > s$, such that either

(1) $$\begin{cases} y+1 = d(r^2 - s^2) \\ 2y = d(2rs) \end{cases}$$

or

(2)
$$\begin{cases} y+1 = d(2rs) \\ 2y = d(r^2-s^2) \end{cases} .$$

For case (1), it follows easily that $d = 1$, while in case (2), $d = 2$. Hence, solving case (1) is equivalent to finding relative prime positive integers r and s of different parity satisfying

(3)
$$\boxed{r^2 - rs - s^2 = 1 .}$$

Now, in case (2), let

$$r' = r+s$$
$$s' = r-s .$$

Then, recalling that $d = 2$ in case (2), we have

(4)
$$\begin{cases} y+1 = r'^2 - s'^2 \\ 2y = 2r's' , \end{cases}$$

which has formally the same appearance as case (1) and implies

$$r'^2 - r's' - s'^2 = 1 .$$

Thus, since

$$r = \frac{r'+s'}{2} \quad \text{and} \quad s = \frac{r'-s'}{2} ,$$

solving case (2) is equivalent to finding odd positive integers r' and s' satisfying (3).

In either case, we see that every solution of (b) with $y > 0$ is generated by an appropriate solution of the diophantine equation:

(*)
$$\boxed{r^2 - rs - s^2 = 1 .}$$

Note that any solution (r, s) of (*) in positive integers has r and s relatively prime and $r > s$. Note that the case (r even, s even) cannot occur as a solution of (*).

Now, if (r, s) is a solution of (*) with positive integers r and s of different parity, then case (1) is indicated with $y = rs$ and either

$x = r^2$ or $x = -s^2$. Thus, we obtain two solutions (x, y) of (b), namely (r^2, rs) and $(-s^2, rs)$.

If (r', s') is a solution of (*) with odd positive integers r' and s', then we have case (2) and $y = r's'$ with both $x = r'^2$ and $x = -s'^2$, again giving two solutions of (b).

Thus, every positive solution (r, s) of (*) leads to two solutions of equation (b) having positive values for y, namely (r^2, rs) and $(-s^2, rs)$.

It remains to consider solutions of (b) having $y < 0$.

If $y < 0$, let $y = -|y|$; then, from (b),

$$x = \frac{(-|y| + 1) \pm \sqrt{(|y| - 1)^2 + 4y^2}}{2} ,$$

so that $(|y| - 1)^2 + 4|y|^2$ must be a perfect square, or equivalently, there exists an integer z such that

$$\boxed{(|y| - 1)^2 + (2|y|)^2 = z^2 .}$$

As before, letting d = the greatest common divisor of $|y| - 1$ and $2|y|$, we deduce the existence of two relatively prime positive integers r and s of different parity, with $r > s$, such that either

(1)* $$|y|-1 = d(r^2-s^2)$$
$$2|y| = d(2rs)$$

or

(2)* $$|y|-1 = d(2rs)$$
$$2|y| = d(r^2-s^2) .$$

Clearly, $d = 1$ in case (1)* and $d = 2$ for case (2)*. In case (1)*, we find that r and s must satisfy

(**) $$\boxed{r^2-s^2-rs = -1 ,}$$

while in case (2)*, the substitution $r' = r+s$, $s' = r-s$ yields (using $d = 1$ for case (1)* and $d = 2$ for case (2)*)

$$|y|-1 = r'^2-s'^2$$
$$2|y| = 2r's' ,$$

which shows that (r', s') is also a solution in positive integers of (**).

Note that any solution (r, s) of (**) in positive integers has r and s relatively prime and $r > s$ if we exclude the solution $r = s = 1$. Also the case (r even, s even) cannot occur as a solution of (**). Thus, every solution of (**) in positive integers either has both terms odd or r even and s odd. The latter case gives a solution of (b) with $|y| = rs$ and both $x = -r^2$ and $x = s^2$, so that the two generated solutions of (b) are $(-r^2, -rs)$ and $(s^2, -rs)$.

Similarly, if (r', s') is a solution of (**) with r' and s' both odd and $r' > s'$, then $|y| = r's'$ with $x = -r'^2$ and $x = s'^2$.

Thus, every solution of (**) in positive integers (r, s) (including $(1, 1)$) yields two solutions of (b) with negative y, namely $(-r^2, -rs)$ and $(s^2, -rs)$.

To find the actual solutions, we recall that every solution of $r^2 - rs - s^2 = 1$ in positive integers r, s has the form $r = F_{2k+1}$ and $s = F_{2k}$ for some integer $k \geq 1$. (See solution of H-31). The corresponding solutions of (b) are $(F_{2k+1}^2, F_{2k} F_{2k+1})$ and $(-F_{2k}^2, F_{2k} F_{2k+1})$ for $k = 1, 2, 3, \ldots$.

The other equation $r^2 - rs - s^2 = -1$ may be transformed to $r'^2 - r's' - s'^2 = 1$ by the change of variable, $r' = r + s$, $s' = r$; it follows that every solution of $r^2 - rs - s^2 = -1$ in positive integers (r, s) has the form $r = F_{2k}$, $s = F_{2k-1}$ for some integer $k \geq 1$. The corresponding solutions of (b) are $(-F_{2k}^2, -F_{2k}F_{2k-1})$ and $(F_{2k-1}^2, -F_{2k} F_{2k-1})$ for $k = 1, 2, 3, \ldots$.

Summarizing, the set of solutions, $(F_{2k+1}^2, F_{2k} F_{2k+1})$, $(-F_{2k}^2, F_{2k} F_{2k+1})$, $(-F_{2k}^2, -F_{2k} F_{2k-1})$, $(F_{2k-1}^2, -F_{2k} F_{2k-1})$ for $k = 1, 2, 3, \ldots$, constitute all the non-zero integral solutions of $x^2 - xy - x - y^2 = 0$, and the set

$$(-F_{2k+1}^2, F_{2k} F_{2k+1}), (F_{2k}^2, F_{2k} F_{2k+1}), (F_{2k}^2, -F_{2k} F_{2k-1}),$$
$$(-F_{2k-1}^2, -F_{2k} F_{2k-1}) \text{ for } k = 1, 2, 3, \ldots$$

constitute all non-zero integral solutions of $x^2 + xy + x - y^2 = 0$.

AN OLD PROBLEM

H-41 *Proposed by Robert A. Laird, New Orleans, La.*

Find rational integers, x, and positive integers, m, so that

$$N = x^2 - m \quad \text{and} \quad M = x^2 + m$$

are rational squares.

Solution by Joseph Arkin, Spring Valley, New York

Professor Oystein Ore, Sterling Professor of Mathematics at Yale University, in his book, Number Theory and Its History, 1st ed., 1948, gives the complete solution to this problem on pages 188-193.

Also solved by Maxey Brooke, Sweeny, Texas

COMMENTS ON THE HISTORICAL CASE

Solved by Robert A. Laird

A solution to the historical problem submitted to Fibonacci (Leonardo of Pisa) by John of Palermo, an imperial notary of Emperor Frederick II, about 1220 A. D. (see page 124, Cajori's "History of Mathematics" for reference). The problem: Find a number x, such that $x^2 + 5$ and $x^2 - 5$ are each square numbers. In other words, find the square which increased or decreased by 5, remains a square. Leonardo solved the problem by a method (not known to me) of building squares by the summation of odd numbers.

Solution to this problem was published in the "Mathematics Teacher" in December 1952.

I offer it here for your interest and pleasure. Let

x = side of the desired square

$x + b$ = side of a larger square

$x - a$ = side of a smaller square

a and b are positive, rational numbers

(1) $$(x + b)^2 = x^2 + 5$$

(2) $$(x^2 - a)^2 = x^2 - 5$$

Solving (1) and (2)

(3) $$x = \frac{5 + a^2}{2a}$$

(4) $$x = \frac{5 - b^2}{2b}$$

Equating (3) and (4)

$$\frac{5 + a^2}{2a} = \frac{5 - b^2}{2b}$$

Solving for b in terms of a, we have

$$b = \frac{-(5+a^2) \pm \sqrt{a^4+30a^2+25}}{2a} \tag{5}$$

In order for b to be a rational number, the radical must clear. So find value of a that will do this.

We can find a by trial substitution or by factoring. Let's take factoring:

$$a^4 + 30a^2 + 25$$

$$a^4 + \underbrace{26a^2 + 4a^2}\ + 25$$

$$a^4 + 26a^2 + 169 + 4a^2 - 144$$

$$(a^2 + 13)^2 + 4(a^2 - 36)$$

If $a^2 = 36$ or $a = 6$, the radical will clear. For immediate result, substitute $a = 6$ in (3)

$$x = \frac{5 + a^2}{2a} = \frac{5 + 36}{12} = \frac{41}{12} \qquad \text{Q. E. D.}$$

Generally, find the square which if increased or decreased by m will remain a square (m = positive integer). Strangely, when m = 6, a solution can be found, but not for m = 1, or 2, or 3, or 4.

FROM BEST SET OF K TO BEST SET OF K+1 ?

H-42 *Proposed by J.D.E. Konhauser, State College, Pa.*

A set of nine integers having the property that no two pairs have the same sum is the set consisting of the nine consecutive Fibonacci numbers, 1, 2, 3, 5, 8, 13, 21, 34, 55 with total sum 142. Starting with 1, and annexing at each step the smallest positive integer which produces a set with the stated property yields the set 1, 2, 3, 5, 8, 13, 21, 30, 39 with sum 122. Is this the best result? Can a set with lower total sum be found?

Partial solution by the proposer.

Partial answer. The set 1, 2, 4, 5, 9, 14, 20, 26, 35 has total sum 116. For eight numbers the best set appears to be 1, 2, 3, 5, 9, 15, 20, 25 with sum 80. Annexing the lowest possible integer to extend the set to nine members requires annexing 38 which produces a set with sum 118. It is not clear (to me, at least) how to progress from a best set of k integers to a best set for k + 1 integers.

H-43 *(Corrected) Proposed by H.W. Gould, West Virginia University, Morgantown, West Va.*

Let

$$\varphi(x) = \sum_{n=1}^{\infty} x^{F_{mn}} ,$$

where F_j is the j-th Fibonacci number, find

$$\lim_{x \to 1} \frac{\varphi(x)}{-\log(1-x)}$$

See special case m = 2 in Revista Matematica Hispano-Americana (2) 9 (1934) 223-225 problem 115.

A FAVORABLE RESPONSE

H-44 *Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let $u_o = q$ and $u_1 = p$, and $u_{n+2} = u_{n+1} + u_n$, then the u_n are called generalized Fibonacci numbers.

(1) Show $$u_n = pF_n + qF_{n-1}$$

(2) Show that if

$$V_{2n+1} = u_n^2 + u_{n+1}^2 \quad \text{and} \quad V_{2n} = u_{n+1}^2 - u_{n-1}^2 ,$$

then V_n are also generalized Fibonacci numbers.

Solution by Lucile R. Morton, San Jose State College, San Jose, California

We prove formula (1) by induction on n. It is obvious that

$$u_1 = p = pF_1 + qF_0 \quad \text{and} \quad u_2 = p + q = pF_2 + qF_1 \quad .$$

Now let us assume formula (1) holds for $n = k$ and $n = k+1$. Thus

$$u_k = pF_k + qF_{k-1}$$

and

$$u_{k+1} = pF_{k+1} + qF_k \quad .$$

Adding we get

$$u_{k+1} + u_k = p(F_{k+1} + F_k) + q(F_k + F_{k-1}) \quad ,$$

or

$$u_{k+2} = pF_{k+2} + qF_{k+1} \quad ,$$

which was to be proved.

We prove V_n are generalized Fibonacci numbers by showing they satisfy the recursion formula $V_{n+2} = V_{n+1} + V_n$, where $V_0 = 2pq - q^2$ and $V_1 = p^2 + q^2$. We can do this by showing

$$V_{2n+1} = V_{2n} + V_{2n-1} \tag{3}$$

$$V_{2n+2} = V_{2n+1} + V_{2n} \quad . \tag{4}$$

From formulas (2)

$$\begin{aligned} V_{2n} + V_{2n-1} &= (u_{n+1}^2 - u_{n-1}^2) + (u_{n-1}^2 + u_n^2) \\ &= u_{n+1}^2 + u_n^2 = V_{2n+1} \quad , \end{aligned}$$

and

$$\begin{aligned} V_{2n+1} + V_{2n} &= (u_n^2 + u_{n+1}^2) + (u_{n+1}^2 - u_{n-1}^2) \\ &= u_n^2 - u_{n-1}^2 + 2u_{n+1}^2 \\ &= (u_{n+1})(u_{n-2}) + (u_{n+1})(u_{n+1} + u_n + u_{n-1}) \\ &= u_{n+1}(u_{n-2} + u_{n-1} + u_{n+2}) \\ &= (u_{n+2} - u_n)(u_{n+2} + u_n) = u_{n+2}^2 - u_n^2 \\ &= V_{2n+2} \quad . \qquad \text{Q. E. D.} \end{aligned}$$

Now let us carry our problem a little further. Let m be a fixed integer, and let $V_n = u_{n+m}$. Are there any restrictions on p and q? Since V_n and u_n are generalized Fibonacci numbers

$$V_{n+1} = V_0F_n + V_1F_{n+1} = (2pq-q^2)F_n + (p^2 + q^2)F_{n+1}$$

and

$$u_{n+m+1} = u_mF_n + u_{m+1}F_{n+1} = (pF_m + qF_{m-1})F_n + (pF_{m+1} + qF_m(F_{n+1} .$$

Thus we have

$$(5) \qquad 2pq-q^2 = pF_m + qF_{m-1}$$

$$(6) \qquad p^2 + q^2 = pF_{m+1} + qF_m$$

Our question becomes: For what integral values p and q do equations (5) and (6) hold? Obviously $p = q = 0$ is a solution. Then $V_n = u_n = 0$. Let

$$p = \frac{x + F_{m+1}}{2} \quad \text{and} \quad q = \frac{y + F_m}{2} ,$$

substituting into equations (5) and (6) we have

$$(7) \qquad 2xy - y^2 = F_{m+1}^2 - F_{m-1}^2 = F_{2m} \quad \text{and}$$

$$(8) \qquad x^2 + y^2 = F_{m+1}^2 + F_m^2 = F_{2m+1} .$$

Eliminating x and simplifying

$$5y^4 + 2(F_{2m} - 2\,F_{2m+1})y^2 + F_{2m}^2 = 0 ,$$

or

$$5y^4 - 2L_{2m}y^2 + F_{2m}^2 = 0 .$$

Thus

$$y^2 = \frac{L_{2m} \pm \sqrt{4L_{2m}^2 - 20F_{2m}^2}}{10} .$$

$$= \frac{L_{2m} \pm \sqrt{L_{2m}^2 - 5F_{2m}^2}}{5} .$$

Then

$$y^2 = \frac{L_{2m} \pm \sqrt{4(-1)^{2m}}}{5} = \frac{L_{2m} \pm 2}{5} = \frac{L_m^2 - 2(-1)^m \pm 2}{5}$$

and we have $5y^2 = L_m^2$, which has no integral solutions, or

$$5y^2 = L_m^2 \pm 4 = 5F_m^2 + 4(-1)^m \pm 4 \quad . \tag{9}$$

Now $5y^2 = 5F_m^2 \pm 8$, which has no integral solutions, or $5y^2 = 5F_m^2$, and $y = \pm F_m$. Therefore the equations (7) and (8) have the solutions $x = F_{m+1}$, $y = F_m$ and $x = -F_{m+1}$, $y = -F_m$ for all m, and $x = -F_{m+1}$, $y = F_m$ and $x = F_{m+1}$, $y = -F_m$ for $m = 0, -1$.

Thus

$$\begin{matrix} p = F_{m+1} \\ q = F_m \end{matrix} \quad \text{or} \quad \begin{matrix} p = 0 \\ q = 0 \end{matrix}$$

are solutions of (5) and (6) for all m, and

$$\begin{matrix} p = 0 \\ q = F_m \end{matrix} \quad \text{or} \quad \begin{matrix} p = F_{m+1} \\ q = 0 \end{matrix}$$

are solutions of (5) and (6) for $m = 0, -1$.

Therefore $V_n = u_{m+n} = F_{2m+n}$ when $p = F_{m+1}$ and $q = F_m$ for all m, or $V_n = u_{m+n} = 0$ when $p = q = 0$.

If we consider nonintegral solutions, from (7) and (8) we had

$$5y^2 = L_m^2$$

which gives us

$$y = \pm \frac{L_m}{\sqrt{5}} \quad \text{and} \quad x = \pm \frac{L_{m+1}}{\sqrt{5}} \quad .$$

Thus the solutions of (7) and (8) are

$$x = \frac{L_{m+1}}{\sqrt{5}}, \quad y = \frac{L_m}{\sqrt{5}} \quad \text{and} \quad x = -\frac{L_{m+1}}{\sqrt{5}}, \quad y = -\frac{L_m}{\sqrt{5}}$$

for all m. Therefore

$$p = \frac{\dfrac{L_{m+1}}{\sqrt{5}} + F_{m+1}}{2} = \frac{\alpha^{m+1}}{\sqrt{5}}$$

$$q = \frac{\dfrac{L_m}{\sqrt{5}} + F_m}{2} = \frac{\alpha^m}{\sqrt{5}}$$

and

$$p = \frac{-\dfrac{L_{m+1}}{\sqrt{5}} + F_{m+1}}{} = -\frac{\beta^{m+1}}{\sqrt{5}}$$

$$q = \frac{-\dfrac{L_m}{\sqrt{5}} + F_m}{2} = -\frac{\beta^m}{\sqrt{5}} \quad .$$

Also solved by Clifton T. Whyburn, Douglas Lind, Clyde A. Bridger, Charles R. Wall, John L. Brown, Jr., Joseph Arkin, Raymond E. Whitney, John Wessner, W.A. Al-Slalm and A. A. Gioia (jointly), Charles Ziegenfus and L. Carlitz.

ITERATED SUMS OF SQUARES

H-45 *Proposed by R.L. Graham, Bell Telephone Labs., Murray Hill, N.J.*

Prove

$$\sum_{p=0}^{n} \sum_{q=0}^{p} \sum_{r=0}^{q} \sum_{s=0}^{r} F_s^2 = F_{n+2}^2 - \frac{1}{8}\left(2n^2+8n+11-3(-1)^n\right) \quad ,$$

where F_n is the nth Fibonacci number.

Solution by Charles R. Wall, Texas Christian University, Ft. Worth, Texas

Using the identities

$$\sum_{n=0}^{k} F_n^2 = \sum_{n=1}^{k} F_n^2 = F_k F_{k+1} \quad ,$$

$$\sum_{n=0}^{k} F_n F_{n+1} = F_{k+1}^2 - \frac{1+(-1)^k}{2} \quad ,$$

we have

$$\sum_{p=0}^{n}\sum_{q=0}^{p}\sum_{r=0}^{q}\sum_{s=0}^{r} F_s^2 = \sum_{p=0}^{n}\sum_{q=0}^{p}\sum_{r=0}^{q} F_r F_{r+1}$$

$$= \sum_{p=0}^{n}\sum_{q=0}^{p}\left(F_{q+1}^2 - \frac{1+(-1)^q}{2}\right)$$

$$= \sum_{p=0}^{n}\left\{F_{p+1}F_{p+2} - \frac{p+1}{2} - \frac{1+(-1)^p}{4}\right\}$$

$$= \sum_{p=0}^{n}\left\{F_{p+1}F_{p+2} - \frac{p}{2} - \frac{3}{4} - \frac{(-1)^p}{4}\right\}$$

$$= F_{n+2}^2 - \frac{1}{2} - \frac{(-1)^n}{2} - \frac{n(n+1)}{4} - \frac{3(n+1)}{4} - \frac{1+(-1)^n}{8}$$

$$= F_{n+2}^2 - \frac{1}{8}\left(2n^2 + 8n + 11 - 3(-1)^n\right) .$$

Also solved by Douglas Lind, L. Carlitz, and Al-Slalm and A. A. Gioia (jointly).

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HAVE YOU SEEN?

J. Arkin, "An Extension of the Fibonacci Numbers," American Mathematical Monthly, Vol. 72, No. 5, March 1965, pp. 275-279.

Marvin Wunderlich, "Another Proof of the Infinite Primes Theorem," American Mathematical Monthly, Vol. 72, No. 5, March 1965, p. 305. This is an extremely neat proof for the Fibonacci Fan!

Benjamin B. Sharpe, Problem 561, Mathematics Magazine, Vol. 28, No. 2, March 1965, pp. 121-122.