ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. HILLMAN
University of Santa Clara, Santa Clara, California

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. Any problem believed to be new in the area of recurrent sequences and any new approaches to existing problems will be welcomed. The proposer should submit each problem with solution in legible form, preferably typed in double spacing with name and address of the proposer as a heading.

Solutions to problems should be submitted on separate sheets in the format used below within two months of publication.

B-64 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $L_n L_{n+1} = L_{2n+1} + (-1)^n$, where L_n is the n-th Lucas number defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$.

B-65 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let u_n and v_n be sequences satisfying $u_{n+2}+au_{n+1}+bu_n=0$ and $v_{n+2}+cv_{n+1}+dv_n=0$ where a, b, c, and d are constants and let $(E^2+aE+b)(E^2+cE+d) = E^4+pE^3+qE^2+rE+s$. Show that $y_n=u_n+v_n$ satisfies

$$y_{n+4}+py_{n+3}+qy_{n+2}+ry_{n+1}+sy_n = 0 \quad .$$

B-66 *Proposed by D.G. Mead, University of Santa Clara, Santa Clara, California*

Find constants p, q, r, and s such that

$$y_{n+4}+py_{n+3}+qy_{n+2}+ry_{n+1}+sy_n = 0$$

is a 4th order recursion relation for the term-by-term products $y_n=u_nv_n$ of solutions of $u_{n+2}-u_{n+1}-u_n = 0$ and $v_{n+2}-2v_{n+1}-v_n = 0$.

B-67 *Proposed by D.G. Mead, University of Santa Clara, Santa Clara, California*

Find the sum $1 \cdot 1+1 \cdot 2+2 \cdot 5+3 \cdot 12+\ldots+F_nG_n$, where $F_{n+2}=F_{n+1}+F_n$ and $G_{n+2}=2G_{n+1}+G_n$.

B-68 *Proposed by Walter W. Horner, Pittsburgh, Pennsylvania*

Find expressions in terms of Fibonacci numbers which will generate integers for the dimensions and diagonal of a rectangular parallelopiped, i.e., solutions of

$$a^2+b^2+c^2 = d^2 \quad .$$

B-69 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Solve the system of simultaneous equations:

$$xF_{n+1}+yF_n = x^2+y^2$$

$$xF_{n+2}+yF_{n+1} = x^2+2xy$$

where F_n is the n-th Fibonacci number.

SOLUTIONS

CHEBYSHEV POLYNOMIALS

B-27 *Proposed by D.C. Cross, Exeter, England*

Corrected and restated from Vol. 1, No. 4: The Chebyshev Polynomials $P_n(x)$ are defined by $P_n(x) = \cos(n\text{Arccos } x)$. Letting $\phi = \text{Arccos } x$, we have

$$\cos\phi = x = P_1(x),$$

$$\cos(2\phi) = 2\cos^2\phi - 1 = 2x^2 - 1 = P_2(x),$$

$$\cos(3\phi) = 4\cos^3\phi - 3\cos\phi = 4x^3 - 3x = P_3(x),$$

$$\cos(4\phi) = 8\cos^4\phi - 8\cos^2\phi + 1 = 8x^4 - 8x^2 + 1 = P_4(x), \quad \text{etc.}$$

It is well known that

$$P_{n+2}(x) = 2xP_{n+1}(x) - P_n(x) \quad .$$

Show that

$$P_n(x) = \sum_{j=0}^{m} B_{jn}x^{n-2j}$$

where

$$m = \left[n/2 \right] ,$$

the greatest integer not exceeding $n/2$, and

(1) $B_{on} = 2^{n-1}$

(2) $B_{j+1,n+1} = 2B_{j+1,n} - B_{j,n-1}$

(3) If $S_n = |B_{on}| + |B_{1n}| + \ldots + |B_{mn}|$, then $S_{n+2} = 2S_{n+1} + S_n$.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

By De Moivre's Theorem,

$$(\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi \quad .$$

Letting $x = \cos \phi$, and expanding the left side,

$$\cos n\phi + i \sin n\phi = (x + i \sqrt{1 - x^2})^n$$

$$= \sum_{j=0}^{n} (-1)^{j/2} \binom{n}{j} x^{n-j} (1 - x^2)^{j/2} \quad .$$

We equate real parts, noting that only the even terms of the sum are real,

$$\cos n\phi = P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} x^{n-2k}(1 - x^2)^k \quad .$$

We may prove from this (cf. Formula (22), p. 185, Higher Transcendtal Functions, Vol. 2 by Erdelyi et al; R. G. Buschman, "Fibonacci Numbers, Chebyshev Polynomials, Generalizations and Difference Equations," Fibonacci Quarterly, Vol. 1, No. 4, p. 2) that

(*) $$B_{j,n} = \frac{n(-1)^j 2^{n-2j-1} (n-j-1)!}{j!\,(n-2j)!} \quad .$$

From this, we have

(1) $$B_{o,n} = 2^{n-1} \quad .$$

It is also easy to show from (*) that

(2) $$B_{j+1,\,n+1} = 2\,B_{j+1,\,n} - B_{j,\,n-1} \quad .$$

Now (*) implies

$$B_{j,\,n} = (-1)^j \left|B_{j,\,n}\right| \quad ,$$

so that (2) becomes

$$(-1)^{j+1} \left|B_{j+1,\,n+1}\right| = 2\,(-1)^{j+1} \left|B_{j+1,\,n}\right| + (-1)^{j+1} \left|B_{j,\,n-1}\right| \quad ,$$

or

$$\left|B_{j+1,\,n+1}\right| = 2\,\left|B_{j+1,\,n}\right| + \left|B_{j,\,n-1}\right| \quad .$$

Summing both sides for j to $\left[\frac{n+1}{2}\right]$, we have

(3) $$S_{n+1} = 2\,S_n + S_{n-1} \quad .$$

Also solved by the proposer.

A SPECIAL CASE

B-52 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $F_{n-2}F_{n+2} - F_n^2 = (-1)^{n+1}$, where F_n is the n-th Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$.

Solution by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

Identity XXII (Fibonacci Quarterly, Vol. 1, No. 2, April 1963, p. 68) states:

$$F_nF_m - F_{n-k}F_{m+k} = (-1)^{n-k}\,F_kF_{m+k-n} \quad .$$

The proposed identity is immediate on taking $m = n$ and $k = 2$.

More generally, we have

$$F_n^2 - F_{n-k}F_{n+k} = (-1)^{n-k}F_k^2 \quad \text{for} \quad 0 \le k \le n \quad .$$

Also solved by Marjorie Bicknell, Herta T. Freitag, John E. Homer, Jr., J.A.H. Hunter, Douglas Lind, Gary C. MacDonald, Robert McGee, C.B.A. Peck, Howard Walton, John Wessner, Charles Ziegenfus, and the proposer.

SUMMING MULTIPLES OF SQUARES

B-53 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that

$$(2n-1)F_1^2 + (2n-2)F_2^2 + \dots + F_{2n-1}^2 = F_{2n}^2 \quad .$$

Solution by James D. Mooney, University of Notre Dame, Notre Dame, Indiana

Remembering that

$$\sum_{k=0}^{n} F_k^2 = F_n F_{n+1} \quad ,$$

we may proceed by induction. Clearly for $n = 1$, $F_1^2 = 1 = F_2^2$. Assume

$$\left[2(n-1) - 1\right] F_1^2 + \left[2(n-1) - 2\right] F_2^2 + \dots + F_{2(n-1)-1} =$$

$$= (2n-3)F_1^2 + (2n-4)F_2^2 + \dots + F_{2n-3} = F_{2n-2}^2 \quad .$$

Then

$$(2n-1)F_1^2 + \dots + F_{2n-1} = \left[(2n-3)F_1^2 + \dots + F_{2n-3}\right] +$$

$$2(F_1^2 + \dots + F_{2n-2}^2) + F_{2n-1}^2 = F_{2n-2}^2 + \sum_{k=0}^{2n-2} F_k^2 + \sum_{k=0}^{2n-1} F_k^2 =$$

$$F_{2n-2}^2 + F_{2n-2}F_{2n-1} + F_{2n-1}F_{2n} = F_{2n-2}^2 + F_{2n-2}F_{2n-1} +$$

$$+ F_{2n-1}(F_{2n-2} + F_{2n-1}) = F_{2n-2}^2 + 2F_{2n-2}F_{2n-1} + F_{2n-1}^2 =$$

$$(F_{2n-2} + F_{2n-1})^2 = F_{2n}^2 \quad . \quad \text{Q.E.D.}$$

Also solved by Marjorie Bicknell, J.L. Brown, Jr., Douglas Lind, John E. Homer, Jr., Robert McGee, C.B.A. Peck, Howard Walton, David Zeitlin, Charles Ziegenfus, and the proposer.

RECURRENCE RELATION FOR DETERMINANTS

B-54 *Proposed by C.A. Church, Jr., Duke University, Durham, N. Carolina*

Show that the n-th order determinant

$$f(n) = \begin{vmatrix} a_1 & 1 & 0 & 0 & & 0 & 0 \\ -1 & a_2 & 1 & 0 & & 0 & 0 \\ 0 & -1 & a_3 & 1 & & 0 & 0 \\ 0 & 0 & -1 & a_4 & \cdots & 0 & 0 \\ \cdots & & & & & & \\ \cdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & a_n \end{vmatrix}$$

satisfies the recurrence $f(n) = a_n f(n-1) + f(n-2)$ for $n > 2$.

Solution by John E. Homer, Jr., La Crosse, Wisconsin

Expanding by elements of the n-th column yields the desired relation immediately.

Also solved by Marjorie Bicknell, Douglas Lind, Robert McGee, C.B.A. Peck, Charles Ziegenfus, and the proposer.

AN EQUATION FOR THE GOLDEN MEAN

B-55 *From a proposal by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Show that $x^n - xF_n - F_{n-1} = 0$ has no solution greater than a, where $a = (1 + \sqrt{5})/2$, F_n is the n-th Fibonacci number, and $n > 1$.

Solution by G.L. Alexanderson, University of Santa Clara, California

For $n > 1$ let $p(x, n) = x^n - xF_n - F_{n-1}$, $g(x) = x^2 - x - 1$, and $h(x, n) = x^{n-2} + x^{n-3} + 2x^{n-4} + \dots + F_k x^{n-k-1} + \dots + F_{n-2}x + F_{n-1}$. It is easily seen that $p(x, n) = g(x)h(x, n)$, $g(x) < 0$ for $-1/a < x < a$, $g(a) = 0$, $g(x) > 0$ for $x > a$, and $h(x, n) > 0$ for $x \geq 0$. Hence $x = a$ is the unique positive root of $p(x, n) = 0$.

Also solved by J.L. Brown, Jr., Douglas Lind, C.B.A. Peck, and the proposer.

GOLDEN MEAN AS A LIMIT

B-56 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Let F_n be the n-th Fibonacci number. Let $x_0 \geq 0$ and define $x_1, x_2, \ldots$ by $x_{k+1} = f(x_k)$ where

$$f(x) = \sqrt[n]{F_{n-1} + xF_n} .$$

For $n > 1$, prove that the limit of x_k as k goes to infinity exists and find the limit. (See B-43 and B-55.)

Solution by G.L. Alexanderson, University of Santa Clara, Santa Clara, California

For $n > 1$ let $p(x) = x^n - xF_n - F_{n-1}$. Let $a = (1 + \sqrt{5})/2$. As in the proof of B-55, one sees that $p(x) > 0$ for $x > a$ and that $p(x) < 0$ for $0 \leq x < a$. If $x_k > a$, we then have

$$(x_k)^n > x_k F_n + F_{n-1} = (x_{k+1})^n$$

and so $x_k > x_{k+1}$. It is also clear that $x_k > a$ implies

$$(x_{k+1})^n = x_k F_n + F_{n-1} > aF_n + F_{n-1} = a^n$$

and hence $x_{k+1} > a$. Thus $x_o > a$ implies $x_o > x_1 > x_2 > \ldots > a$. Similarly, $0 \leq x_o < a$ implies $0 \leq x_o < x_1 < x_2 < \ldots < a$. In both cases the sequence $x_o, x_1, \ldots$ is monotonic and bounded. Hence x_k has a limit $L > 0$ as k goes to infinity. Since L satisfies

$$L = \sqrt[n]{F_{n-1} + LF_n} ,$$

L must be the unique positive solution of $p(x) = 0$.

Also solved by Douglas Lind and the proposer.

A FIBONACCI-LUCAS INEQUALITY

B-57 *Proposed by G.L. Alexanderson, University of Santa Clara, Santa Clara, California*

Let F_n and L_n be the n-th Fibonacci and n-th Lucas number respectively. Prove that

$$(F_{4n}/n)^n > L_2 L_6 L_{10} \cdots L_{4n-2}$$

for all integers $n > 2$.

Solution by David Zeitlin, Minneapolis, Minnesota

Using mathematical induction, one may show that

$$F_{4n} = \sum_{k=1}^{n} L_{4k-2}, \quad n = 1, 2, \ldots \quad .$$

If we apply the well-known arithmetic-geometric inequality to the unequal positive numbers $L_2, L_6, L_{10}, \ldots, L_{4n-2}$, we obtain for $n = 2, 3, \ldots,$

$$\frac{F_{4n}}{n} = \frac{\sum_{k=1}^{n} L_{4k-2}}{n} = \sqrt[n]{L_2 L_6 L_{10} \cdots L_{4n-2}} \quad ,$$

which is the desired inequality.

Also solved by Douglas Lind and the proposer.

xxxxxxxxxxxxxxx

ACKNOWLEDGMENT

It is a pleasure to acknowledge the assistance furnished by Prof. Verner E. Hoggatt, Jr. concerning the essential idea of "Maximal Sets" and the line of proof suggested in the latter part of my article "On the Representations of Integers as Distinct Sums of Fibonacci Numbers." The article appeared in Feb.,1965. H. H. Ferns

CORRECTION Volume 3, Number 1

Page 26, line 10 from bottom of page

$$V_{7,3}+V_{7,4}+V_{7,5} = F_8-F_7 = F_6 = 8$$

Page 27, lines 4 and 5

$$F_2+F_4+F_6+\ldots+F_n = F_{n+1} -1 \quad \text{(n even)}$$

$$F_3+F_5+F_7+\ldots+F_n = F_{n+1} -1 \quad \text{(n odd)}$$

ACKNOWLEDGMENT

Both the papers "Fibonacci Residues" and "On a General Fibonacci Identity," by John H. Halton, were supported in part by NSF grant GP2163.

CORRECTION Volume 3, Number 1

Page 40, Equation (81), the R. H. S. should have an additional term

$$- v^2 F_{v+2}$$