ON IDENTITIES INVOLVING FIBONACCI NUMBERS

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Rather extensive lists of identities involving Fibonacci numbers have been given by K. Subba Rao [1] and by David Zeitlin [2]. Additional identities are presented here, with the feature that summation by parts has been used for effecting the proofs (except for identity 23).

Let $f_o = 0$ and $f_1 = 1$ and let $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Then

(1) $$\sum_{k=0}^{n} kf_k = nf_{n+2} - f_{n+3} + 2$$

(2) $$\sum_{k=0}^{n} (1)^k kf_k = (-1)^n(n+1)f_{n-1} + (-1)^{n-1}f_{n-2} - 2, \; n \geqq 2$$

(3) $$\sum_{k=0}^{n} (-1)^k f_{2k} = \left[(-1)^n(f_{2n+2} + f_{2n}) - 1 \right] /5$$

(4) $$\sum_{k=0}^{n} (-1)^k f_{2k+1} = \left[(-1)^n(f_{2n+3} + f_{2n+1}) + 2 \right] /5$$

(5) $$\sum_{k=0}^{n} kf_{2k} = (n+1)f_{2n+1} - f_{2n+2}$$

(6) $$\sum_{k=0}^{n} kf_{2k+1} = (n+1)f_{2n+2} - f_{2n+3} + 1$$

(7) $$\sum_{k=0}^{n} (-1)^k kf_{2k} = (-1)^n(nf_{2n+2} + (n+1)f_{2n})/5$$

(8) $$\sum_{k=0}^{n} (-1)^k kf_{2k+1} = (-1)^n(nf_{2n+3} + (n+1)f_{2n+1})/5 - 1/5$$

(9) $$\sum_{k=0}^{n} k^2 f_{2k} = (n^2+2)f_{2n+1} - (2n+1)f_{2n} - 2$$

(10) $$\sum_{k=0}^{n} k^2 f_{2k+1} = (n^2+2)f_{2n+2} - (2n+1)f_{2n+1} - 1$$

(11) $$\sum_{k=0}^{2n} kf_k^2 = (2n+1)f_{2n}f_{2n+1} - f_{2n+1}^2 + 1$$

(12) $$2\sum_{k=0}^{n} (-1)^k kf_{m+3k} = (-1)^n(n+1)f_{m+3n+1}$$
$$- ((-1)^n f_{m+3n+2} + f_{m-1})/2, \quad m=2, 3, \ldots$$

(13) $$3\sum_{k=0}^{n} (-1)^k kf_{m+4k} = (-1)^n(n+1)f_{m+4n+2}$$
$$- ((-1)^n f_{m+4n+4} + f_m)/3, \quad m=2, 3, \ldots$$

(14) $$121\sum_{k=0}^{n} (-1)^k kf_{m+5k} = (-1)^n \left[(55n+35)f_{m+5n+1}\right.$$
$$\left. - 25f_{m-5n+2} + (22n+18)f_{m+5n}\right]$$
$$- \left[20f_{m+1} - 17f_m - 10f_{m-1}\right], \quad m=1, 2, \ldots$$

(15) $$\sum_{k=0}^{n} \sum_{k_1=0}^{k} f_{k_1} = f_{n+4} - (n+3)$$

(16) $$\sum_{k=0}^{n} k \sum_{k_1=0}^{k} f_{k_1} = (n+1)f_{n+4} - f_{n+6} + 5 - n(n+1)/2$$

(17) $$\sum_{k=0}^{n} k^2 f_k = (n^2 + 2)f_{n+2} - (2n-3)f_{n+3} - 8$$

(18) $$\sum_{k=0}^{n} k^3 f_k = (n^3 + 6n - 12)f_{n+2} - (3n^2 - 9n + 19)f_{n+3} + 50$$

(19)
$$\sum_{k=0}^{n} k^4 f_k = (n^4 + 12n^2 - 48n + 98)f_{n+2}$$
$$-(4n^3 - 18n^2 + 76n - 159)f_{n+3} - 416$$

(20)
$$\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$$

(21)
$$\sum_{k=0}^{n} f_k^2 f_{k+1} = \frac{1}{2} f_{n+2} f_{n+1} f_n$$

(22)
$$\sum_{k=0}^{n} f_k^2 f_{k+2} = \frac{1}{2} (f_{n+3} f_{n+1} f_n - (-1)^n f_{n-1} + 1)$$

(23)
$$\sum_{k=0}^{n} f_k^3 = \frac{1}{2} (f_{n+1}^2 f_n - (-1)^n f_{n-1} + 1) \quad n \geq 1$$

(24)
$$\sum_{k=0}^{n} kf_k^3 = \frac{n+1}{2} (f_{n+1}^2 f_n - (-1)^n f_{n-1}) - \frac{1}{4} f_{n+2} f_{n+1}^2$$
$$+ \frac{(-1)^n}{4}(3f_n - 2f_{n-1}) + \frac{5}{4}$$

The well-known method of summation by parts is established from the identity

$$u_k \Delta v_k = \Delta (u_k v_k) - v_{k+1} \Delta u_k$$

On summing there results

$$\sum_{k=0}^{n} u_k \Delta v_k = u_k v_k \Big|_0^{n+1} - \sum_{k=0}^{n} v_{k+1} \Delta u_k$$

Of course, a suitable choice of u_k and Δv_k is essential just as it is in integration by parts. In order to find v_k from Δv_k results in [1] and [2] have been used when needed. Also, any constant term in v_k can be omitted in the two terms of the right member.

To prove (1), let $u_k = k$ and $\Delta v_k = f_k$. Then $\Delta u_k = 1$ and

$$v_k = \sum_{i=0}^{k-1} f_i = f_{k+1} - 1 .$$

Omitting the constant -1 from v_k, we find

$$\sum_{k=0}^{n} kf_k = kf_{k+1}\Big|_0^{n+1} - \sum_{k=0}^{n} 1 \cdot f_{k+2} = (n+1)f_{n+2} - (f_{n+4} - 1 - f_1)$$

$$= nf_{n+2} - (f_{n+4} - f_{n+2}) + 2$$

$$= nf_{n+2} - f_{n+3} + 2$$

To prove (2), let $u_k = k$ and

$$\Delta v_k = (-1)^k f_k = \sum_{i=0}^{k} (-1)^i f_i - \sum_{i=0}^{k-1} (-1)^i f_i .$$

Then $\Delta u_k = 1$ and $v_k = (-1)^k f_{k-2} - 1$. Omitting the term -1 from v_k, with $k \geq 2$

$$\sum_{k=0}^{n} (-1)^k kf_k = \sum_{k=2}^{n} (-1)^k kf_k - 1 = k(-1)^{k-1} f_{k-2}\Big|_2^{n+1} - \sum_{k=2}^{n} (-1)^k f_{k-1} - 1$$

$$= (-1)^n (n+1) f_{n-1} + \sum_{k=1}^{n-1} (-1)^k f_k - 1$$

$$= (-1)^n (n+1) f_{n-1} + (-1)^{n-1} f_{n-2} - 2$$

To prove (3) and (4), together, write in (3) $u_k = (-1)^k$ and

$$\Delta v_k = \sum_{i=0}^{k} f_{2i} - \sum_{i=0}^{k-1} f_{2i}$$

So that $\Delta u_k = 2(-1)^{k-1}$ and $v_k = f_{2k-1}$. Then

$$A = \sum_{k=0}^{n} (-1)^k f_{2k} = \sum_{k=1}^{n} (-1)^k f_{2k} = (-1)^k f_{2k-1}\Big|_1^{n+1} - 2\sum_{k=0}^{n} (-1)^{k+1} f_{2k+1} - 2$$

$$= (-1)^{n+1} f_{2n+1} - 1 + 2B,$$

where

$$B = \sum_{k=0}^{n} (-1)^k f_{2k+1}.$$

In (4) let $u_k = (-1)^k$ and $\Delta v_k = f_{2k+1}$ so $\Delta u_k = 2(-1)^{k+1}$ and $v_k = f_{2k}$. Then

$$B = \sum_{k=0}^{n} (-1)^k f_{2k+1} = (-1)^k f_{2k}\Big|_0^{n+1} - 2\sum_{k=0}^{n} (-1)^{k+1} f_{2k+2}$$

$$= (-1)^{n+1} f_{2n+2} + 2(-1)^n f_{2n+2} - 2A$$

Solving gives the results.

To obtain (5) let $u_k = k$ and then $v_k = f_{2k-1}$. This gives

$$\sum_{k=0}^{n} kf_{2k} = kf_{2k-1}\Big|_0^{n+1} - \sum_{k=0}^{n} f_{2k+1} = (n+1)f_{2n+1} - f_{2n+2}$$

The others are proved similarly, except that (23) was obtained from (21) and (22). Note that the same method could be used to extend the results.

REFERENCES

1. K. Subba Rao, "Some Properties of Fibonacci Numbers," American Mathematical Monthly, Vol. 60, 1953, pp. 680-684.
2. David Zeitlin, "On Identities for Fibonacci Numbers," American Mathematical Monthly, Vol. 70, 1963, pp. 987-991.

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