A PERMUTATIVE PROPERTY OF CERTAIN MULTIPLES OF THE NATURAL NUMBERS

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1. INTRODUCTION

In number theory one encounters such numbers as

$$.105263157894736842$$

(the period of 2/19) and .102564 (the period of 4/39) one of whose very interesting properties will be treated here. If the terminal digit be removed from the end of the number and placed at the beginning, the result is the product of that digit and the original number.

Examples:

$$\begin{array}{r} .10526315789473684\underline{2} \\ \times\ 2 \\ \hline .\underline{2}10526315789473684 \end{array} \quad \text{and} \quad \begin{array}{r} .10256\underline{4} \\ \times\ 4 \\ \hline .\underline{4}10256 \end{array}$$

The purpose of this paper will be to investigate the existence and characteristics of such numbers.

2. DEFINITIONS

A positive number G will be called a gauntlet if it has a cyclic permutation with the property that, when the natural number g making up its last n digits be moved to the first n digits' positions of the number, then the result is exactly the product gG. When such a number G exists for a natural number g we will occasionally write G(g) for emphasis. The product gG is called the second order gauntlet, written $G^{(2)}$.

We also define the function D whose value D(x) is the number of digits in x. It follows from the above definitions that $D(G) = D(G^{(2)})$.

3. FAMILY OF GAUNTLETS

The question arises: are there many gauntlets for a single natural number? We answer with a theorem.

Theorem 1. Each natural number for which a gauntlet exists has infinitely many gauntlets each consisting of a number of sets of the same period.

Proof. Let $.p_1p_2\ldots p_{D(G)}$ be a digit-wise representation of G, a gauntlet of the natural number g. We observe that gG is of the form $.q_1q_2\ldots q_{D(g)}$ because $D(G^{(2)}) = D(G)$. This means there is no carry on the left after multiplication of G by g. This implies

$$g\cdot(.p_1\ldots p_{D(G)}p_1\ldots p_{D(G)}) = .q_1\ldots q_{D(G)}q_1\ldots q_{D(G)}$$

and the theorem follows by induction.

Example:

$$g = 4$$

$$G_1(g) = .10256\underline{4}$$

$$G_1^{(2)}(g) = .\underline{4}10256$$

$$G_2(g) = .10256410256\underline{4}$$

$$G_2^{(2)} = .\underline{4}10256410256$$

Let us call numbers which are gauntlets for the same natural number and whose digits are repetitions of the digits of a simpler gauntlet members of the family of that gauntlet. Similarly we define a family of second order gauntlets. Hereafter unless otherwise stated G and $G^{(2)}$ will be understood to be the least positive gauntlets of their families.

4. DIGITS COMMON TO ALL GAUNTLETS

Theorem 2. The leading non-zero digit of a gauntlet is 1.

Proof. Let g be represented by the digit-wise expansion $c_1c_2\ldots c_{D(g)}$. Then $G^{(2)} = .c_1c_2\ldots c_{D(g)}x_1x_2\ldots x_{D(G)-D(g)}$. Now

$$(1)\qquad c_1\ldots c_{D(g)}\overline{\big)\,.c_1\ldots c_{D(g)-1}c_{D(g)}x_1x_2\ldots x_{D(G)-D(g)}}^{\;.0\;\ldots 0\qquad 1\qquad\ldots}$$

and by definition the quotient must be G.

Corollary 2. A gauntlet of the natural number g has exactly $D(g)-1$ leading zeros.

Proof. Count the leading zeros of the quotient of (1).

Note. The leading zeros are part of the repeating set of digits in the family of a gauntlet.

Theorem 3. For g not a power of 10 there are exactly $2D(g)-1$ zeros to the immediate right of the leading non-zero digit 1 of G.

Proof. From (1)

$$G = .0_1 \cdots 0_{D(g)-1} 1 x_1 \cdots x_{D(G)-D(g)}$$

(the x_i are now the unknown digits of the numerator) where

$$x_{D(G)-2D(g)+1} \cdots x_{D(G)-D(g)} = c_1 \cdots c_{D(g)} = g.$$

Whence

$$G^{(2)} = .c_1 \cdots c_{D(g)} 0_1 \cdots 0_{D(g)-1} 1\ x_1 x_2 \cdots x_{D(G)-2D(g)}.$$

Then by definition

$$g\,\overline{\big)\,.c_1 \cdots c_{D(g)-1} c_{D(g)} 0_1 \cdots 0_{D(g)-1} 1 \quad x_1 x_2 \cdots x_{D(G)-2D(g)}}^{\;.0_1 \cdots 0_{D(g)-1} 1 \quad 0_1 \cdots 0_{D(g)-1} 0_{D(g)} \;\cdots}$$

which implies

$$G = .0_1 \cdots 0_{D(g)-1} 1\ 0_1 \cdots 0_{D(g)-1} 0_{D(g)} \cdots$$

This means that

$$g\,\overline{\big)\,.c_1 \cdots c_{D(g)-1} c_{D(g)} 0_1 \cdots 0_{D(g)-1} 1 \quad 0_1 \;\cdots\; 0_{D(g)-1}\ 0_{D(g)} \cdots}^{\;.0_1 \cdots 0_{D(g)-1} 1 \quad 0_1 \cdots 0_{D(g)-1} 0_{D(g)} 0_{D(g)+1} \cdots 0_{2D(g)-1} x \cdots}$$

(and x is non-zero because $10_1 \cdots 0_{D(g)}$ is greater that g) which proves the theorem.

Corollary 3. The gauntlet of a natural number g which is a power of 10 is exactly $.0_1 \cdots 0_{D(g)-1} 1\ 0_1 \cdots 0_{D(g)-1}$.

Proof. That $g=10^n$ implies $D(g)=n+1$. That is to say $g=10_1 \cdots 0_n = 10_1 \cdots 0_{D(g)-1}$, the terminal $D(g)$ digits of

$$.0_1 \cdots 0_{D(g)-1} 1\ 0_1 \cdots 0_{D(g)-1},$$

and

$$\begin{array}{r} .0_1\cdots 0_{D(g)-1}1\ 0_1\cdots 0_{D(g)-1} \\ \times\ \ 1\ 0_1\cdots 0_{D(g)-1} \\ \hline .10_1\cdots 0_{D(g)-1}0_1\cdots 0_{D(g)-1} \end{array} \qquad \text{Q. E. D.}$$

Exceptions must always be made in the following discussion for $g=10^n$ because only with such a g are the $D(g)$ initial digits of g^2 the digits of g itself.

Examples for the corollary.

$$G(1) = .1$$

$$G(10) = .010$$

It should be obvious by now that it is largely inconsequential whether we consider gauntlets as integers or decimals, because whether the number is 010 or .010 the digits are the same and our primary concern is which leading or trailing zeros are part of the number, not where the decimal point goes. It is more amenable to the notion of families to use decimals because of the obvious similarity to periodic decimals. However, in a following theorem (Theorem 5) the proof is expedited by reference to gauntlets as integers.

5. GENERATION OF A GAUNTLET IN SETS OF DIGITS

Let us now examine the interrelationships of the digits within a gauntlet and the way in which a natural number generates its own gauntlet.

Remark. The following discussion develops an algorithm which finds G for $g \neq 10^n$. Corollary 3 found G for every $g=10^n$, and it may be readily verified that the algorithm of this section finds a larger member of the family of $G(10^n)$.

The terminal $D(g)$ digits of G make up g itself. Consequently the terminal $D(g)$ digits of $G^{(2)}$ must be the terminal $D(g)$ digits of g^2 which are also the $D(g)+1$st through the $2D(g)$th digits of G, counting from the righthand side. That is,

$$G = .x_{D(G)}\cdots x_{2D(g)+1}d_{2D(g)}\cdots d_{D(g)+1}c_{D(g)}\cdots c_1$$

where the d's are the $D(g)$ terminal digits of g^2 and of $gG=G^{(2)}$. Moving leftward along G we see that the next set of $D(g)$ x's must represent the terminal $D(g)$ digits of the sum of the leading digits of g^2 not included in the set $d_{D(g)} \dots d_1$ and $g \cdot (d_{D(g)} \dots d_1)$. So is the next set of $D(g)$ digits related to those to the right of it. To restate symbolically what we have just verbalized, the ith set of $D(g)$ digits (counting from the right where the a's are the sets) may be written

$$(2) \qquad a_i = ga_{i-1} + r_{i-1} - \left[\frac{ga_{i-1}+r_{i-1}}{10^{D(g)}}\right] \cdot 10^{D(g)}$$

where

$$(3) \qquad r_i = \left[\frac{a_{i-1}g}{10^{D(g)}}\right], \quad a_1 = g, \quad \text{and} \quad r_1 = 0.$$

(Brackets indicate greatest integer division.)

These equations, which follow directly from the definitions, constitute an algorithm which, depending upon g alone, inevitably produces $G(g)$ if it exists. Since the algorithm generates only sets of $D(g)$ digits each we may conclude $D(g)$ divides $D(G)$ and when G exists it has a left-most set a_j whose digit-wise representation is $0 \dots 01$ and that $r_{j+1}=0$. These conditions provide criteria for stopping the algorithm at a_j.

Remark. The single exception to the rule "$D(g)$ divides $D(G)$" is for $g=10^n$. The reason is that the two a_i of $G(10^n)$ share the common digit 1. However, the algorithm will find a $G'(10^n) > G$ such that $D(g)$ divides $D(G')$. That $G(10^n)$ is the only possible exception for the success of the algorithm may be readily verified.

Theorem 4. If G exists for a given g the algorithm (given above) generates G, and the condition $a_j=1$ and $r_{j+1}=0$ is sufficient to terminate the algorithm.

Proof. That the algorithm generates G follows from the preceding remarks in this section. If $a_j=1$ and $r_{j+1}=0$ the algorithm begins to repeat the digits of G because $a_{j+1}=g \cdot 1+0-0=g$, and $r_{j+2}=0$. This is identically the situation at the beginning of the algorithm, which means from this point it would regenerate the same digits. Hence if a_j is the first set equal to 1 and such that $r_{j+1}=0$ then the digits generated up to that point make up the least positive member of the family, that is G.

Remark. An algorithm mentioned by Johnson [2] will find the period of the reciprocal of 10m-1 (where m is a natural number), but the result does not have the combined multiplicative and permutative property, which is the subject of this paper, for m of more than one digit.

Example. The period .10027, a cyclic permutation of that found for m=37 by Johnson's method, has not the same property as has the number found by my method for m=37, namely

$$\begin{array}{r} .01000\ 27034\ 33360\ 36766\ 69\underline{37} \\ \times\ 37 \\ \hline .\underline{37}010\ 00270\ 34333\ 60367\ 6669 \end{array}$$

6. THE EXISTENCE THEOREM

Theorem 5. For every natural number there exists at least one gauntlet and hence one family of the gauntlet.

Proof. That $G(10^n)$ exists follows from Corollary 3. Assume $g \neq 10^n$. As usual we assume G is the smallest positive member of its family. We recall that D counts all the digits in a number which are part of that number. This includes leading zeros. Let G be considered an integer. The relationship between g and G, from the definitions, is

$$\frac{G-g}{10^{D(g)}} + g10^{D(G)-D(g)} = gG = g^{(2)}$$

which simplifies thus:

$$G-g+10^{D(G)}g = 10^{D(g)}gG$$

$$G(1-10^{D(g)}g) + g(10^{D(G)}-1) = 0$$

$$G = \frac{g(10^{D(G)}-1)}{10^{D(g)}g-1}$$

Now we require that G be an integer, which is true if and only if $g(10^{D(G)}-1)$ is congruent to 0 modulo $10^{D(g)}g-1$. This means

$$10^{D(G)}g \equiv g \bmod(10^{D(g)}g-1).$$

Since $10^{D(g)}g-1$ and g are relatively prime

$$10^{D(G)} \equiv 1 \bmod(10^{D(g)}g-1).$$

Now

$$(4) \qquad 10^{x} \equiv 1 \bmod(10^{D(g)}g-1)$$

has a solution $x = \phi(10^{D(g)}g-1)$ by Fermat's theorem because 10 and $10^{D(g)}g-1$ are relatively prime. That is to say

$$(5) \qquad 10^{x}g \equiv g \bmod(10^{D(g)}g-1)$$

has a solution which means there exists an integer K such that

$$(6) \qquad K = \frac{g(10^{x}-1)}{10^{D(g)}g-1}$$

for a given integer g.

All solutions to (4) may be found in the following way. We divide successively increasing powers of 10 by $10^{D(g)}g-1$ until finally we are left with a remainder of 1. This implies the solution to (5) may be found similarly. We divide the product of g and successively increasing powers of 10 by $10^{D(g)}g-1$ until finally there is a remainder of g. The number of zeros we use is the solution x.

Now (6) has a least positive solution x_0. Let the numerator (7) of the following expression be the least positive such numerator, that is let the appearance of g as a remainder be the first such appearance of g. If we can show that (7) is G we are finished since $D((7))$ which is x_0 will also be $D(G)$, and x_0 is known to be the least positive solution of (6) such that K is the least positive integer, and G is assumed to be the least positive gauntlet of g.

(7)
$$10^{D(g)}g-1 \;\overline{)g.0\;0\;\ldots 0}\quad .p_1p_2\ldots p_{x_0}$$
$$\vdots$$

(8) $$\overline{q_1\ldots 0}$$
(9) $$\ldots$$
(10) $$\overline{\qquad g}$$

For keeping track of our zeros we will revert to the use of decimals. Adding terminal zeros to 1.000... is simplified by the nature of the number (i.e. 1.0 followed by infinitely many zeros is equivalent to 1.0). We find ourselves studying

$$\frac{1}{g10^{D(g)}-1}$$

or, equivalently,

$$\frac{g}{g10^{D(g)}-1}$$

as far as x_0 is concerned, rather than

$$\frac{10^{x_0}}{g10^{D(g)}-1} \quad \text{or} \quad \frac{g10^{x_0}}{g10^{D(g)}-1}\ .$$

Let g be expanded digitwise as $c_1\ldots c_{D(g)}$. Since $10^{D(g)}g-1$ ends in 9, and (8) ends in 0 while g ends in $c_{D(g)}$, then p_{x_0} can only be $c_{D(g)}$. We rewrite (8), (9) and (10) as (13), (14) and (15) below:

(11) $$\ldots$$
(12) $$\ldots$$
(13) $$q_1 \;\ldots\; 0$$
(14) $$c_{D(g)}(g10^{D(g)}-1)$$
(15) $$c_1 \;\ldots\; c_{D(g)}\ .$$

We introduce the convention of braces about the digit-wise expansion of a number to clarify arithmetic expressions. Then we may write (14) as

$$\{c_1\ldots c_{D(g)}\}\cdot 10^{D(g)}c_{D(g)}-c_{D(g)}\ .$$

Adding g we have (13):

$$\{c_1 \cdots c_{D(g)}\} \cdot c_{D(g)} 10^{D(g)} - c_{D(g)} + \{c_1 \cdots c_{D(g)}\}$$

which reduces to

$$\{c_1 \cdots c_{D(g)}\} \cdot c_{D(g)} 10^{D(g)} + \{c_1 \cdots c_{D(g)-1}\} \cdot 10.$$

But (13) without the suffixed 0 is

$$\{c_1 \cdots c_{D(g)}\} \cdot c_{D(g)} 10^{D(g)-1} + \{c_1 \cdots c_{D(g)-1}\}$$

which terminates in $c_{D(g)-1}$. This means that

$$p_{x_0-1} = c_{D(g)-1}, \text{ whence (12) is } (g10^{D(g)}-1) \cdot c_{D(g)-1}.$$

This implies that (11) is

$$\{c_1 \cdots c_{D(g)}\} \cdot c_{D(g)-1} 10^{D(g)} - c_{D(g)-1}$$
$$+ \{c_1 \cdots c_{D(g)}\} \cdot c_{D(g)} 10^{D(g)-1} + \{c_1 \cdots c_{D(g)-1}\} .$$

Redcuing as before and removing the suffixed 0 we have for (11)

$$\{c_1 \cdots c_{D(g)}\} \cdot \{c_{D(g)-1} c_{D(g)}\} \cdot 10^{D(g)-2} + \{c_1 \cdots c_{D(g)-2}\} .$$

By induction after $D(g)$ such steps the remainder is

(16) $$\{c_1 \cdots c_{D(g)}\} \quad \{c_1 \cdots c_{D(g)}\} \cdot 10^0 + \{0\} .$$

At each step the terminal digit in the remainder was a c_i. This implies

$$p_{x_{0-D(g)+1}} \cdots p_{x_0} = c_1 \cdots c_{D(g)}.$$

At this point the remainder ends in $\langle g^2 \rangle$. (The new notation means the last digit of.) This means

$$p_{x_0 - D(g)} = \langle g^2 \rangle .$$

This seems to indicate generation of the same digits of the algorithm of section 5. Indeed they are identical because the minuend producing the remainder (16) is

$$\{c_1 \dots c_{D(g)}\} \cdot 10^{D(g)} \langle g^2 \rangle - \langle g^2 \rangle + g^2$$

which after removal of the suffixed zero is

$$\{c_1 \dots c_{D(g)}\} \quad \langle g^2 \rangle \, 10^{D(g)-1} + \frac{g^2 - \langle g^2 \rangle}{10}$$

which ends in $\langle g^2 - \langle g^2 \rangle \rangle$, and we see we must exhaust $D(g)$ powers of 10 again, thereby setting $p_{x_0-2D(g)+1} \dots p_{x_0-D(g)}$ equal to the terminal $D(g)$ digits of g^2.

Alternatively we must, every $D(g)$ steps, exhaust the $D(g)$ digits of a set which corresponds to some a_i of the algorithm. Therefore by Theorem 4 the numerator is G if its first $D(g)$ digits are $0_1 \dots 0_{D(g)-1} 1$ and its next $D(g)$ digits are 0. This latter condition is sufficient to make $r_{i+1} = 0$.

We write the initial situation in the division process as

$$\begin{array}{r|l}
 & .\,0_1 \dots 0_{D(g)-1} 1 \;\dots \\
\{c_1 \dots c_{D(g)}\} \cdot 10^{D(g)} - 1 & \overline{c_1 \dots c_{D(g)} .\, 0_1 \dots 0_{D(g)-1} 0_{D(g)} \dots} \\
 & \underline{\{c_1 \dots c_{D(g)} \; 0_1 \quad \cdot \quad \cdot \quad 0_{D(g)}\} - 1} \\
 & \hfill 1
\end{array}$$

because

$$\{c_1 \dots c_{D(g)}\} \cdot 10^{D(g)} = c_1 \dots c_{D(g)} 0_1 \dots 0_{D(g)}$$

and since

$$10^{2D(g)-1} \le g10^{D(g)} - 1 < 10^{2D(g)}$$

we have

$$\begin{array}{r|l}
 & .\,0_1 \dots 0_{D(g)-1} 1 \quad 0_1 \dots 0_{2D(g)-1} \dots \\
\{c_1 \dots c_{D(g)}\} \cdot 10^{D(g)} - 1 & \overline{c_1 \dots c_{D(g)} .\, 0_1 \dots 0_{D(g)-1} 0_{D(g)} 0_{D(g)+1} \dots}\;.
\end{array}$$

Q. E. D.

Corollary 5. For every natural number there is only one family of gauntlets and only one G, the least positive gauntlet.

Proof. The uniqueness of the algorithmic process and also of the division in the previous theorem.

7. ADDITIONAL THEOREMS

The following theorems, which may be easily verified, are submitted without proof.

Theorem 6. The period of $n/(n10^{D(n)}-1)$ where n is any positive integer is the same as the period of the reciprocal of $n10^{D(n)}-1$.

Theorem 7. Each digit of the period on $n/(n10^{D(n)}-1)$ appears in succession as the terminal digit of a remainder when decimal division is carried out.

Example:

$$g = 4$$
$$D(g) = .10256\underline{4}$$
$$g10^{D(g)}-1 = 39$$

```
      .102564
39 )4.000000
    39
    ①0
    00
    1⓪0
     78
     2②0
     195
      2⑤0
      234
       1⑥0
       154
         ④
```

Theorem 8. The digits of the period of $1/(n10^{D(n)}-1)$ are a cyclic permutation leftward $D(g)$ places of those of $n/(n10^{D(n)}-1)$ where n is any natural number, and theorem 7 holds for $1/(n10^{D(n)}-1)$.

Theorem 9. For G the gauntlet of a given g, the following relation holds, $2D(g10^{D(g)}-1) \leq D(G) \leq g10^{D(g)}-2$.

Theorem 10. $D(g)$ divides the period of $g/(g10^{D(g)}-1)$ and hence of $1/(g10^{D(g)}-1)$, provided $g \neq 10^n$, and, for $g=10^n$, then $D(G) = 2D(g)-1$.

8. PARTIAL TABLE OF THE FIRST 100 GAUNTLETS

			The Period of a permutation	
g	G	D(G)	of	of
1	.1	1	$\frac{1}{9}$	$\frac{1}{9}$
2	.10526 31578 94736 842	18	$\frac{2}{19}$	$\frac{1}{19}$
3	.10344 82758 62068 96551 72413 793	28	$\frac{3}{29}$	$\frac{1}{29}$
4	.10256 4	6	$\frac{4}{39}$	$\frac{1}{39}$
7	.10144 92753 62318 84057 97	22	$\frac{7}{69}$	$\frac{1}{69}$
34	.01000 29420 41776 99323 33039 12915 5634	34	$\frac{34}{3399}$	$\frac{1}{3399}$
37	.01000 27034 33360 36766 6937	24	$\frac{37}{3699}$	$\frac{1}{3699}$
100	.00100	5	$\frac{100}{99999}$	$\frac{1}{99999}$

9. APPENDIX

An interesting question is, are there any more integers, g, such as 1 and 34, where D(G) = g?

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