ON THE EQUATIONS $U_n = U_q x^2$, WHERE q IS ODD, AND $V_n = V_q x^2$, WHERE q IS EVEN

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1. Introduction

Let $\{w_n\}$ be the sequence satisfying the second-order linear recurrence

(1.1) $\quad w_n = pw_{n-1} + w_{n-2}, \ n \in \mathbb{Z},$

where w_0, w_1 are given integers and p is an *odd* positive integer.

Of particular interest are the generalized Fibonacci and Lucas sequences, $\{U_n(p)\}$ and $\{V_n(p)\}$, respectively, which are defined by (1.1) and the initial conditions

$$U_0(p) = 0, \ U_1(p) = 1,$$

and

$$V_0(p) = 2, \ V_1(p) = p.$$

Cohn [2] has proved the two theorems below, which we shall need later.

Theorem 1: The equation $V_n(p) = x^2$ has:

(1) if $p = 1$, two solutions $n = 1, 3$;
(2) if $p = 3$, one solution $n = 3$;
(3) if $p \neq 1$ is a perfect square, one solution $n = 1$;
(4) no solution otherwise.

The equation $V_n(p) = 2x^2$ has the solution $n = 0$, and for a finite number of values of p also $n = \pm 6$, but no other solutions.

Theorem 2: The equation $U_n(p) = x^2$ has:

(1) the solutions $n = 0$, and $n = \pm 1$;
(2) if p is a perfect square, the solution $n = 2$;
(3) if $p = 1$, the solution $n = 12$,
(4) no other solutions.

Recently, Goldman [3] has shown that if $L_n = L_{2^m} x^2$, where L_{2^m} is prime, then $n = \pm 2^m$. Adapting Cohn's and Goldman's method, we shall prove here the following theorems.

Theorem A; Let $q \geq 2$ be an *even* integer. Then $V_n(p) = V_q(p)x^2$, if and only if $n = \pm q$.

Theorem B: Let $q \geq 3$ be an *odd* integer. Then the equation $U_n(p) = U_q(p)x^2$ has the solutions

(1) $n = 0$, and $n = \pm q$,
(2) if $p = 1$ or 3, $q = 3$, and $n = 6$,

and no other solutions.

2. Preliminaries

The following formulas are well known (see [1], [4], [5]) or easily proved (recall that p is odd). For the sake of brevity, we shall write U_n and V_n, instead of $U_n(p)$ and $V_n(p)$.

(a) $U_{-n} = (-1)^{n+1}U_n$, and $V_{-n} = (-1)^n V_n$,
(b) $U_{2n} = U_n V_n$,
(c) if $d = \gcd(m, n)$, then $U_d = \gcd(U_m, U_n)$,
(d) if $q \geq 3$, then $U_q | U_n$ iff $q | m$,
(e) if $q \geq 2$, then $V_q | V_n$ iff $q | n$, and n/q is odd,
(f) if an *odd* prime number divides V_q and V_k, then $v_2(q) = v_2(k)$, where $v_2(s)$ is the 2-adic value of the integer s,
(g) $2 | V_n$ iff $3 | n$,
(h) if $k \equiv \pm 2 \pmod 6$, then $V_k \equiv 3 \pmod 4$,
(i) $\gcd(U_n, V_n) = 1$ or 2,
(j) if $\{w_n\}$ is a sequence satisfying (1.1), then, for all integers n, k,

$$w_{n+2k} + (-1)^k w_n = w_{n+k} V_k.$$

The following fundamental lemma (see [2], [3]) is recalled here with a new proof.

Lemma 1: If $\{w_n\}$ is a sequence satisfying (1.1), and k an even number, then, for all integers n, t

$$w_{n+2kt} \equiv (-1)^t w_n \pmod{V_k}.$$

Proof: By (j) we have, since k is even

$$w_{n+2k} \equiv -w_n \pmod{V_k},$$

and the proof follows by induction upon t. Q.E.D.

We shall also need the next result.

Lemma 2: If q and k are integers, with q odd and $k \equiv \pm 2 \pmod 6$, then

$$\gcd(U_q, V_k) = 1.$$

Proof: By (h) and (i), notice that $\gcd(U_k, V_k) = 1$, since V_k is odd. Let

$$d = \gcd(q, k) = \gcd(q, 2k).$$

By (b) and (c), we have

$$\gcd(U_q, V_k) | \gcd(U_q, U_{2k}) = U_d,$$

and $U_d | U_k$, since $d | k$. Thus,

$$\gcd(U_q, V_k) | U_k,$$

and so $\gcd(U_q, V_k) = 1$, since $\gcd(U_k, V_k) = 1$. Q.E.D.

3. Proofs of Theorems

Proof of Theorem A: Assume that $V_n = V_q x^2$, where $q \geq 2$ is even, and $n \neq \pm q$. Since $V_q | V_n$, it follows from (e) that

$$\begin{aligned} n &= (\pm 1 + 4j)q, \quad j \neq 0 \\ &= \pm q + 2.3^r k, \end{aligned}$$

where $2jq = 3^r k$, and $k \equiv \pm 2 \pmod 6$. By Lemma 1 and (a),

$$V_n \equiv -V_{\pm q} = -V_q \pmod{V_k},$$

since q is even; hence,

$$-V_q \equiv V_q x^2 \pmod{V_k}.$$

Since $2jq = 3^r k$, then $v_2(k) > v_2(q)$, so by (f) and (g), $\gcd(V_q, V_k) = 1$ since V_k is odd; hence,

$$-1 \equiv x^2 \pmod{V_k},$$

which is impossible, since $V_k \equiv 3 \pmod 4$. Q.E.D.

Proof of Theorem B: Assume that $U_n = U_q x^2$, where $q \geq 3$ is odd, and $n \neq \pm q$. Since $U_q | U_n$, it follows from (d) that $q | n$.

Assume first that n is even, $n = 2jq$, and note that $j \geq 1$, since n even and negative would imply that $U_n < 0$. By (b), we get

$$U_{jq} V_{jq} = U_q x^2;$$

hence,

$$V_{jq} = y^2 \quad \text{or} \quad V_{jq} = 2y^2,$$

since $U_q | U_{jq}$ and $\gcd(U_{jq}, V_{jq}) = 1$ or 2.

If $j = 1$, then $V_q = y^2$ or $V_q = 2y^2$, which imply by Theorem 1 that $p = 1$ or 3, and $q = 3$, $n = 6$; it can be verified that

$$U_6(1) = U_3(1).2^2 \quad \text{and} \quad U_6(3) = U_3(3).6^2.$$

If $j \geq 2$, then $V_{jq} = y^2$ must be rejected by Theorem 1, since $jq > 3$ and $V_{jq} = 2y^2$ can be satisfied only if $jq = 6$, by Theorem 2, i.e., for $q = 3$, $j = 2$, and $n = 12$. However,

$$U_{12} = U_3 x^2$$

can be written, by (b),

$$U_3 V_3 V_6 = U_3 x^2 \quad \text{or} \quad V_3 V_6 = x^2.$$

Since $V_6 = 2y^2$, then $V_3 = 2z^2$, and this is impossible by Theorem 1.

Second, assume that $U_n = U_q x^2$, where n is odd,

$$\begin{aligned} n &= (\pm 1 + 4j)q, \; j \neq 0, \\ &= \pm q + 2.3^r k, \end{aligned}$$

where $k \equiv \pm 2 \pmod 6$. Then, by Lemma 1 and (a),

$$U_n \equiv -U_{\pm q} = -U_q \pmod{V_k},$$

since q is odd. Therefore, by Lemma 2 and hypothesis,

$$-1 \equiv x^2 \pmod{V_k},$$

which is impossible, as above. Q.E.D.

References

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