GCD-CLOSED SETS AND THE DETERMINANTS OF GCD MATRICES

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1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite ordered set of distinct positive integers. The $n \times n$ matrix $[S] = (s_{ij})$, where $s_{ij} = (x_i, x_j)$, the greatest common divisor of x_i and x_j, is called the greatest common divisor (GCD) matrix on S (see [2]). In [6], H. J. S. Smith showed that if S is a factor-closed set, then the determinant of $[S]$, det$[S]$, is $\phi(x_1)\,\phi(x_2)\,\ldots\,\phi(x_n)$, where $\phi(x)$ is Euler's totient function. A set S of positive integers is said to be factor-closed if all positive factors of any member of S belong to S. In [2], we considered GCD matrices in the direction of their structure, determinant, and arithmetic in Z_n, the ring of integers modulo n. In [1], we generalized Smith's result by extending the factor-closed sets to a larger class of sets called gcd-closed sets. A set $S = \{x_1, x_2, \ldots, x_n\}$ as above is said to be gcd-closed if for every i and $j = 1, 2, \ldots, n$, (x_i, x_j) is in S. Every factor-closed set is gcd-closed, but not conversely.

Using structure theorems in [2], Zhongshan Li [4] obtained the value of the determinant of a GCD matrix defined on an arbitrary ordered set of distinct positive integers, and proved the converse of Smith's result. Since the formula derived in [4] is valid for any GCD matrix, it also solves the problem stated in [5] for arithmetic progressions.

In this paper we shall provide another formula for the determinant of a GCD matrix based on the class of gcd-closed sets. Li's formula comes as a corollary. We also use this new formula to find closed-form expressions for the determinants of some special GCD matrices.

2. Preliminary Results

It was remarked in [2] that the determinant of the GCD matrix defined on a set S is independent of the order of the elements of S. Thus, if $S = \{x_1, x_2, \ldots, x_n\}$, we may henceforth assume that $x_1 < x_2 < \cdots < x_n$. Given this natural order on S, we let $B(x_i)$ denote the sum

$$B(x_i) = \sum_{\substack{d \mid x_i \\ d \nmid x_t \\ t < i}} \phi(d),$$

for all $i = 1, 2, \ldots, n$. We note that $B(x_i) = \phi(x_i)$ for all i if and only if S is factor-closed.

The following proposition can be found in [1].

Proposition A: Let $S = \{x_1, x_2, \ldots, x_n\}$ be gcd-closed with $x_1 < x_2 < \cdots < x_n$. Then, for every i and $j = 1, 2, \ldots, n$,

$$(x_i, x_j) = \sum_{x_k \mid (x_i, x_j)} B(x_k).$$

It is clear that any set S of positive integers is contained in a gcd-closed set. By $\overline{S}$ we mean the minimal such gcd-closed set, or *gcd-closure* of S.

It is worthwhile to observe that $\overline{S}$ usually contains considerably fewer elements than any factor-closed set containing S. We now prove a structure theorem for GCD matrices.

Theorem 1: Let $\overline{S} = \{x_1, x_2, \ldots, x_m\}$ be the gcd-closure of $S = \{y_1, y_2, \ldots, y_n\}$ with $x_1 < x_2 < \cdots < x_m$ and $y_1 < y_2 < \cdots < y_n$. Then $[S]$ is the product of an $n \times m$ matrix A and the incidence matrix C corresponding to the transpose of A.

Proof: Define $A = (a_{ij})$ via

$$a_{ij} = \begin{cases} B(x_j) & \text{if } x_j \text{ divides } y_i, \\ 0 & \text{otherwise.} \end{cases}$$

If we let $C = (c_{ij})$ be the incidence matrix corresponding to the transpose of A, then the (i, j)-entry of AC is equal to

$$\sum_{k=1}^{n} a_{ik}c_{kj} = \sum_{\substack{x_k | y_i \\ x_k | y_j}} a_{ik} = \sum_{x_k | (y_i, y_j)} B(x_k),$$

which is equal to (y_i, y_j) by Proposition A and the fact that $\overline{S}$ is gcd-closed.

Remark 1: In the above theorem, $\overline{S}$ may actually be replaced with any gcd-closed set containing S.

The following corollaries appeared in [1].

Corollary 1: If $S = \{x_1, x_2, \ldots, x_n\}$ is gcd-closed with $x_1 < x_2 < \cdots < x_n$, then

$$\det[S] = B(x_1)B(x_2) \ldots B(x_n).$$

Corollary 2 (Smith): If $S = \{x_1, x_2, \ldots, x_n\}$ is factor-closed, then

$$\det[S] = \phi(x_1)\phi(x_2) \ldots \phi(x_n).$$

Corollary 3: Let $S = \{x_1, x_2, \ldots, x_n\}$ be gcd-closed. Then

$$\det[S] = \phi(x_1)\phi(x_2) \ldots \phi(x_n)$$

if and only if S is factor-closed.

Remark 2: It was actually shown in [4] that the converse of Corollary 2 is true.

3. The Value of det$[S]$

The (i, j)-entry of the matrix A in Theorem 1 may be written as $e_{ij}B(x_j)$, where $e_{ij} = 1$ if x_j divides y_i, and 0 otherwise. Let E be the $n \times m$ matrix (e_{ij}). Thus, $C = E^{\mathsf{T}}$, the transpose of E. If Λ is the $m \times m$ diagonal matrix with diagonal $(B(x_1), B(x_2), \ldots, B(x_m))$, we have that $AC = E\Lambda E^{\mathsf{T}}$.

Now let $k_1, k_2, \ldots, k_n$ be distinct positive integers such that

$$1 \le k_1 < k_2 < \cdots < k_n \le m,$$

and let $E_{(k_1, k_2, \ldots, k_n)}$ denote the submatrix of E consisting of the k_1th, ..., k_nth columns of E. Define $A_{(k_1, \ldots, k_n)}$ similarly. It is clear that

$$\det A_{(k_1, \ldots, k_n)} = B(x_{k_1})B(x_{k_2}) \ldots B(x_{k_n}) \cdot \det E_{(k_1, \ldots, k_n)},$$

since

$$A_{(k_1, \ldots, k_n)} = E_{(k_1, \ldots, k_n)} \cdot D,$$

where D is the $n \times n$ diagonal submatrix of Λ with diagonal $(B(x_{k_1}), \ldots, B(x_{k_n}))$.

The following theorem gives the value of $\det[S]$ in terms of $B(x_1)$, $B(x_2)$, $\ldots$, $B(x_m)$.

Theorem 2: Let S and $\overline{S}$ be as in Theorem 1. Then $\det[S]$ is given by the sum

$$\sum_{1 \le k_1 < k_2 < \cdots < k_n \le m} (\det E_{(k_1, \ldots, k_n)})\, B(x_{k_1}) \ldots B(x_{k_n}).$$

Proof: From Theorem 1, $[S] = AC$. Now apply the Cauchy-Binet formula (see [3], p. 22) to obtain

$$\det[S] = \det(AC) = \sum_{1 \le k_1 < k_2 < \cdots < k_n \le m} \det A_{(k_1, \ldots, k_n)} \cdot \det(E_{(k_1, \ldots, k_n)})^T;$$

the result follows from the preceding remarks.

Corollary 4 (Li [4], Theorem 2): Let S be as in Theorem 1 and let $S^* = \{x_1, x_2, \ldots, x_m\}$ be the minimal factor-closed set containing S, with $x_1 < x_2 < x_3 < \cdots < x_m$. Then

$$\det[S] = \sum_{1 \le k_1 < k_2 < \cdots < k_n \le m} (\det E_{(k_1, \ldots, k_n)})^2 \phi(x_{k_1}) \ldots \phi(x_{k_n}).$$

Remark 3: By using a proof similar to that occurring in Li's paper for the converse of Corollary 2 (see [4], Theorem 3), one may establish the converse of Corollary 1.

4. Determinants of Special Matrices

Although the matrices $E_{(k_1, \ldots, k_n)}$ in Theorem 2 are (0, 1)-matrices, it is not true in general that $\det E_{(k_1, \ldots, k_n)} = \pm 1$. In this section, we consider certain sets S which have the property that every such submatrix $E_{(k_1, \ldots, k_n)}$ has determinant equal to 1 or -1, and thus find a closed-form expression for $\det[S]$.

A set $S = \{x_1, x_2, \ldots, x_n\}$ is said to be a k-set if $(x_i, x_j) = k$ for every $i, j = 1, 2, \ldots, n$. For example, {6, 9, 15, 21, 33} is a 3-set. Let S be a k-set. Then either $\overline{S} = S \cup \{k\}$ or $\overline{S} = S$.

Case 1. If $x_1 < x_2 < \cdots < x_n$ and $k = x_1$, then S is gcd-closed, and $B(x_i) = x_i - k$ for $i = 2, 3, \ldots, n$. Hence, by Corollary 1,

$$\det[S] = k(x_2 - k) \ldots (x_n - k).$$

Case 2. Suppose $k \ne x_1$ so that $\overline{S} = \{k = x_0, x_1, x_2, \ldots, x_n\}$. By Theorem 2,

$$\det[S] = \sum_{0 \le t_1 < t_2 < \cdots < t_n \le n} (\det E_{(t_1, \ldots, t_n)})^2 B(x_{t_1}) B(x_{t_2}) \ldots B(x_{t_n}).$$

Lemma 1: $\det E_{(t_1, \ldots, t_n)} = \pm 1$.

Proof: If $(t_1, \ldots, t_n) = (0, 2, 3, \ldots, n)$ or $(1, 2, 3, \ldots, n)$, then $E_{(t_1, \ldots, t_n)}$ is a lower triangular matrix with diagonal $(1, 1, \ldots, 1)$. Thus, $\det E_{(t_1, \ldots, t_n)} = 1$. If

$$(t_1, \ldots, t_n) = (0, 1, \ldots, s-1, s+1, \ldots, n) \text{ for } 2 \le s \le n,$$

then Row s of $E_{(t_1, \ldots, t_n)}$ is $(1, 0, 0, \ldots, 0)$. Moreover, the submatrix of $E_{(t_1, \ldots, t_n)}$ formed by removing Column 1, i.e.,

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

and Row s is the $(n-1)\times(n-1)$ identity matrix. Hence,

$$\det E_{(t_1,\ldots,t_n)} = \pm 1.$$

This completes the proof.

Now $B(x_0) = k$ and $B(x_i) = x_i - k$ for $i > 0$. Thus, by Theorem 2,

$$\det[S] = k\cdot\left(\sum_{i=1}^{n}\frac{(x_1-k)\cdots(x_n-k)}{(x_i-k)}\right) + (x_1-k)\cdots(x_n-k).$$

Cases 1 and 2 above may therefore be combined into the following theorem.

Theorem 3: If $S = \{x_1, x_2, \ldots, x_n\}$ is a k-set with $x_1 < x_2 < \cdots < x_n$, then

$$\det[S] = k(x_2-k)\cdots(x_n-k) + [k(x_1-k)\cdots(x_n-k)]\left[\frac{1}{k}+\frac{1}{x_2-k}+\cdots+\frac{1}{x_n-k}\right].$$

Corollary 5: Let $S = \{x_1, x_2, \ldots, x_n\}$ consist of pairwise coprime positive integers. If $x_1 < x_2 < \cdots < x_n$, then

$$\det[S] = (x_2-1)\cdots(x_n-1) + [(x_1-1)\cdots(x_n-1)]\left[1+\frac{1}{x_2-1}+\cdots+\frac{1}{x_n-1}\right].$$

Corollary 6: Let $p_1, p_2, \ldots, p_n$ be primes with $p_1 < p_2 < \cdots < p_n$. If $S = \{p_1, p_2, \ldots, p\}$, then

$$\det[S] = (p_1-1)\cdots(p_n-1)\left[1+\frac{1}{p_1-1}+\cdots+\frac{1}{p_n-1}\right]$$
$$= \phi(p_1)\cdots\phi(p_n)\left[1+\frac{1}{\phi(p_1)}+\cdots+\frac{1}{\phi(p_n)}\right].$$

Finally, in view of Lemma 1, and for lack of a counterexample, we make the following conjecture and leave it as a problem.

Conjecture: Let S and $\bar{S}$ be as in Theorem 3, with $n > 3$. If $\det E_{(k_1,k_2,\ldots,k_n)} = \pm 1$ for every choice of $k_1, k_2, \ldots, k_n$, then either S is gcd-closed or S is a k-set for some positive integer k.

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