ON REPRESENTATIONS OF NUMBERS BY SUMS OF TWO TRIANGULAR NUMBERS

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1. Introduction

We begin our discussion with a definition.

Definition: As usual,

$$Z := \{0, \pm 1, \pm 2, \ldots\},\ N := \{0, 1, 2, \ldots\},\ P := N\backslash\{0\}.$$

Then, for each $n \in N$,

$$r_2(n) := |\{(x, y) \in Z^2 \mid n = x^2 + y^2\}|,$$
$$t_2(n) := |\{(x, y) \in N^2 \mid n = x(x+1)/2 + y(y+1)/2\}|.$$

Also, for each $n \in P$ and each $i \in \{1, 3\}$,

$$d_i(n) = \sum_{\substack{d \mid n \\ d \equiv i \pmod 4}} 1.$$

We can now state two theorems.

Theorem 1 (Jacobi): For each $n \in P$,

$$r_2(n) = 4\{d_1(n) - d_3(n)\}.$$

Theorem 2: For each $n \in N$,

$$t_2(n) = d_1(4n+1) - d_3(4n+1).$$

Clearly, $r_2(0) = t_2(0) = 1$. Next, we observe that, for positive integers, Theorem 2 can be deduced from Theorem 1. In this note we give an independent proof of Theorem 2. Our proof is based on the triple-product identity

$$(1) \qquad \prod_1^\infty (1 - x^{2n})(1 - ax^{2n-1})(1 - a^{-1}x^{2n-1}) = \sum_{-\infty}^{\infty} (-1)^n x^{n^2} a^n,$$

which is valid for each pair of complex numbers a, x such that $a \neq 0$ and $|x| < 1$. Hirschhorn [2] showed how to deduce Jacobi's theorem from the triple-product identity. The reader will doubtless note that our method is similar to that of Hirschhorn.

2. Proof of Theorem 2

Separating even and odd terms on the right side of (1), and then again using (1) to replace the series in the resulting identity by infinite products, we get

$$\prod_1^\infty (1 - x^{2n})(1 - ax^{2n-1})(1 - a^{-1}x^{2n-1})$$
$$= \sum_{-\infty}^{\infty} x^{4n^2} a^{2n} - ax \sum_{-\infty}^{\infty} x^{4n(n+1)} a^{2n}$$
$$= \prod_1^\infty (1 - x^{8n})(1 + a^2x^{8n-4})(1 + a^{-2}x^{8n-4})$$
$$- (a + a^{-1})x \prod_1^\infty (1 - x^{8n})(1 + a^2x^{8n})(1 + a^{-2}x^{8n}).$$

With D_a denoting derivation with respect to a, we then operate on both sides of the foregoing identity with aD_a to get

$$\begin{aligned}(2)\quad & -\prod_1^\infty (1-x^{2n})(1-ax^{2n-1})(1-a^{-1}x^{2n-1})\sum_1^\infty v_k(x)(a^k-a^{-k}) \\ &= 2\prod_1^\infty (1-x^{8n})(1+a^2x^{8n-4})(1+a^{-2}x^{8n-4})\sum_1^\infty (-1)^{k-1}v_k(x^4)(a^{2k}-a^{-2k}) \\ &\quad -(a-a^{-1})x\prod_1^\infty (1-x^{8n})(1+a^{-2}x^{8n})(1+a^{-2}x^{8n}) \\ &\quad -(a+a^{-1})2x\prod_1^\infty (1-x^{8n})(1+a^2x^{8n})(1+a^{-2}x^{8n})\sum_1^\infty (-1)^{k-1}u_k(x^8)(a^{2k}-a^{-2k}),\end{aligned}$$

where, for convenience $u_k(x) := x^k \cdot (1-x^k)^{-1}$, $v_k(x) := x^k \cdot (1-x^{2k})^{-1}$, $k \in P$, and x is a complex number with $|x| < 1$. Now, in (2), let $a = i$ and divide the resulting identity by $-2i$ to get

$$\prod_1^\infty (1-x^{2n})(1+x^{4n-2})\sum_0^\infty (-1)^k v_{2k+1}(x) = x\prod_1^\infty (1-x^{8n})^3,$$

or, equivalently,

$$x\prod_1^\infty \frac{(1-x^{8n})^3}{(1-x^{2n})(1+x^{4n-2})} = \sum_0^\infty (-1)^k \frac{x^{2k+1}}{1-x^{4k+2}}.$$

Hence,

$$x\prod_1^\infty \frac{(1-x^{8n})^2}{(1-x^{8n-4})^2} = \sum_0^\infty (-1)^k \frac{x^{2k+1}}{1-x^{4k+2}} = \sum_{k=0}^\infty (-1)^k \sum_{j=0}^\infty x^{(2j+1)(2k+1)}.$$

Owing to a well-known identity of Gauss ([1], p. 284), it then follows that

$$\begin{aligned}\sum_0^\infty t_2(n)x^{4n+1} &= x\left\{\sum_0^\infty x^{2n(n+1)}\right\}^2 = x\prod_1^\infty \frac{(1-x^{8n})^2}{(1-x^{8n-4})^2} \\ &= \sum_{k=0}^\infty (-1)^k \sum_{j=0}^\infty x^{(2j+1)(2k+1)} = \sum_{m=0}^\infty x^{2m+1}\sum_{d|2m+1}(-1)^{(d-1)/2} \\ &= \sum_{n=0}^\infty x^{4n+1}\sum_{d|4n+1}(-1)^{(d-1)/2} + \sum_{n=0}^\infty x^{4n+3}\sum_{d|4n+3}(-1)^{(d-1)/2}.\end{aligned}$$

Equating coefficients of like powers of x, we get, for each $n \in \mathbb{N}$,

$$\begin{aligned}t_2(n) = \sum_{d|4n+1}(-1)^{(d-1)/2} &= \sum_{\substack{d|4n+1 \\ d\equiv 1 \pmod 4}} 1 - \sum_{\substack{d|4n+1 \\ d\equiv 3 \pmod 4}} 1 \\ &= d_1(4n+1) - d_3(4n+1),\end{aligned}$$

$$\sum_{d|4n+3}(-1)^{(d-1)/2} = 0.$$

This proves Theorem 2. In passing we note that the second conclusion follows easily from the following independent argument. For each $n \in \mathbb{N}$ and each divisor d (and codivisor d') of $4n + 3$, exactly one of the pair (d, d') is $\equiv 1 \pmod 4$ and exactly one is $\equiv 3 \pmod 4$. Hence,

$$(-1)^{(d-1)/2} + (-1)^{(d'-1)/2} = 0.$$

Summing over all of these pairs, we obtain the desired result.

Finally, we prove that Theorems 1 and 2 are actually *equivalent*. To this end, we first recall the following well-known result.

Theorem: For an arbitrary positive integer $n > 1$, let

$$n = \prod_{i=1}^{i=r} p_i^{e_i}$$

denote its prime-power decomposition. Then, n is representable as a sum of two squares if and only if, for each $i \in \{1, 2, \ldots, r\}$ such that $p_i \equiv 3 \pmod 4$, e_i is even.

It then follows that counting representations of positive integers by sums of two squares can be restricted to positive integers of the form $2^f(4k + 1)$, $f, k \in \mathbb{N}$. Equivalence of Theorems 1 and 2 will then be an easy consequence of the following lemma.

Lemma: If for each $k \in \mathbb{N}$,

$$S = S(k) := \{(x, y) \in \mathbb{N} \times \mathbb{P} \mid 4k + 1 = x^2 + y^2\}$$

and

$$T = T(k) := \{(i, j) \in \mathbb{N}^2 \mid k = i(i + 1)/2 + j(j + 1)/2\},$$

then

$$|S| = |T|.$$

Proof: To see this we define a function $\theta: T \to S$ as follows: for each $(i, j) \in T$,

$$\theta(i, j) := \begin{cases} (0,\ 2i + 1), & \text{if } i = j, \\ (i - j,\ i + j + 1), & \text{if } i > j, \\ (i + j + 1,\ j - i), & \text{if } i < j. \end{cases}$$

Simple calculation reveals that θ is single-valued, and always $\theta(i, j) \in S$. So, we proceed to show that θ is one-to-one from T onto S.

Suppose that (g, h), $(i, j) \in T$, and $\theta(g, h) = \theta(i, j)$. If (a) $g = h$, then

$$\theta(g, h) := (0, 2g + 1).$$

Therefore, $\theta(i, j) \in \mathbb{N} \times \mathbb{P}$ must also have first coordinate equal to 0: that is, $\theta(i, j) = (0, y)$, with $i = j$ and $y = 2i + 1$. So, $2g + 1 = 2i + 1$, whence $g = i$, whence $g = h = i = j$, whence $(g, h) = (i, j)$. If (b) $g > h$, then

$$\theta(g, h) := (g - h, g + h + 1).$$

Therefore, $\theta(i, j) = (x, y) \in \mathbb{P}^2$, with $x < y$, whence $x = i - j$ and $y = i + j + 1$, whence $i - j = g - h$ and $i + j + 1 = g + h + 1$, whence $(i, j) = (g, h)$. If (c) $g < h$, then

$$\theta(g, h) := (g + h + 1, h - g).$$

As before, we must have:

$$g + h = i + j \quad \text{and} \quad -g + h = -i + j,$$

whence $(g, h) = (i, j)$. Thus, θ is one-to-one.

Pick any $(x, y) \in S(k)$, and split two cases: (i) $x = 0$ or (ii) $x > 0$. Under (i) we have

$$4k + 1 = 0^2 + y^2, \text{ whence } y = 2i + 1, \text{ for some } i \in \mathbb{N}.$$

Hence, for $i = j := (y - 1)/2$, we have

$$(x, y) = (0, 2i + 1) = \theta(i, j), \text{ where } (i, j) \in T(k).$$

Under case (ii) we split two further subcases: (ii′) $x < y$ or (ii″) $x > y$. Then under (ii′) we put $i - j = x$ and $i + j + 1 = y$ to find

$$i = (x + y - 1)/2 \quad \text{and} \quad j = (-x + y - 1)/2.$$

Thus, $i > j$, $i - j = x$, and $i + j + 1 = y$, whence $(x, y) = \theta(i, j)$. [Clearly, $(i, j) \in T(k)$.] Under (ii″) we put $i + j + 1 = x$ and $-i + j = y$ to find

$$i = (x - y - 1)/2 \quad \text{and} \quad j = (x + y - 1)/2.$$

As before, $i < j$ and $(x, y) = \theta(i, j)$, where $(i, j) \in T(k)$. This proves that θ is onto S.

Now let us assume that Theorem 2 holds. Then, for each $k \in \mathbf{N}$,

$$|S(k)| = |T(k)| = d_1(4k + 1) - d_3(4k + 1).$$

Therefore,

$$\begin{aligned} r_2(4k + 1) &= |\{(x, y) \in \mathbf{Z}^2 | 4k + 1 = x^2 + y^2\}| \\ &= 4\{d_1(4k + 1) - d_3(4k + 1)\}, \end{aligned}$$

since each solution $(x, y) \in S$ yields 4 solutions $(\pm x, \pm y) \in \mathbf{Z}^2$.

Conversely, let us assume that Theorem 1 holds. Then, for each $k \in \mathbf{N}$,

$$|S(k)| = r_2(4k + 1)/4 = d_1(4k + 1) - d_3(4k + 1),$$

whence (owing to our Lemma),

$$t_2(k) := |T(k)| = d_1(4k + 1) - d_3(4k + 1),$$

as well.

Since $r_2(2^f(4k + 1)) = r_2(4k + 1)$, equivalence of Theorems 1 and 2 follows.

Owing to the equivalence of the two theorems, our proof of Theorem 2 is a new one for both theorems.

Acknowledgment

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References

1. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 4th ed. Oxford: Clarendon Press, 1960.
2. M. D. Hirschhorn. "A Simple Proof of Jacobi's Two-Square Theorem." *Amer. Math. Monthly* 92 (1985):579-80.

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