STRONG DIVISIBILITY LINEAR RECURRENCES
OF THE THIRD ORDER

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1. Introduction

A k^{th}-order linear recurrent sequence $\mathbf{u} = \{u_n : n = 1, 2, \ldots\}$ of integers, satisfying the following property for greatest common divisors:

$$(u_i, u_j) = |u_{(i,j)}| \quad \text{for all } i, j \geq 1,$$

is called a k^{th}-order strong divisibility sequence (SDS). The notion of strong divisibility was introduced by C. Kimberling in [3] for k^{th}-order linear recurrences $\{u_n : n = 0, 1, 2, \ldots\}$.

All the second-order SDS's have been described in [2]. A characterization of all the SDS's in certain subsystems of the system T of all the third-order linear recurrences of integers was given in [1]. The purpose of this note is to extend the results of [1] and to describe all the SDS's in further subsystems of T.

Let U denote the system of all the sequences $\mathbf{u} = \{u_n : n = 1, 2, \ldots\}$ defined by

$$u_1 = 1,\ u_2 = \nu \neq 0,\ u_3 = \mu \neq 0$$

$$u_{n+3} = a \cdot u_{n+2} + b \cdot u_{n+1} + c \cdot u_n, \quad \text{for } n \geq 1,$$

where ν, μ, a, b, and c are integers. The system of all the strong divisibility sequences from U will be denoted by D.

Notice that we may take $u_1 = 1$ without loss of generality as all the third-order SDS's with $u_2 \neq 0 \neq u_3$ are exactly all the nonzero integral multiples of the sequences from D.

Lemma 1.1: Let $\mathbf{u} = \{u_n\} \in U$. Then $u_2 | u_4$ if and only if there exists an integer f such that

(1) $\quad c = f \cdot \nu - a \cdot \mu.$

Proof: From the above definition we obtain $u_2 = \nu$, $u_4 = a\mu + b\nu + c$ and the assertion follows.

2. The Case $a = b = c = 1$

Let V denote the system of all the sequences from U satisfying the condition $a = b = c = 1$, i.e., $\mathbf{u} = \{u_n\} \in V$ if and only if

$$u_1 = 1,\ u_2 = \nu \neq 0,\ u_3 = \mu \neq 0$$
(2)
$$u_{n+3} = u_{n+2} + u_{n+1} + u_n, \quad \text{for } n \geq 1.$$

The following theorem will show that there are no SDS's in V.

Theorem 2.1: The system of sequences V contains no strong divisibility sequences, i.e., $V \cap D = \emptyset$.

Proof: Let us suppose that $\mathbf{u} = \{u_n\} \in V \cap D$. By Lemma 1.1, there exists an integer f such that

(3) $\quad \mu = f \cdot \nu - 1$

and thus

$$u_4 = \nu \cdot (f + 1).$$

Then by (2):

$$u_5 = \nu \cdot (f + 2) + \mu \quad \text{and} \quad u_6 = \nu \cdot (2f + 3) + 2\mu.$$

From $u_2 | u_6$, $u_3 | u_6$, and $(\nu, \mu) = 1$, we get $\nu | 2$ and $\mu | 2f + 3$. Then, using (3), we obtain:

$$\nu = 1,\ \mu | 5 \quad \text{or} \quad \nu = -1,\ \mu | 1 \quad \text{or} \quad \nu = 2,\ \mu | 4 \quad \text{or} \quad \nu = -2,\ \mu | 2.$$

But ν, μ are coprime, which leaves 10 possible pairs of ν and μ. For all of them it is easy to find i, j (always ≤ 9) such that $(u_i, u_j) \ne |u_{(i,j)}|$. Therefore $\mathbf{u} \notin D$, a contradiction.

3. The Case $\mu = 1$; $a = b = 1$

Let W denote the system of all the sequences from U satisfying the conditions $\mu = 1$; $a = b = 1$, i.e., $\mathbf{u} = \{u_n\} \in W$ if and only if

$$(4) \qquad \begin{aligned} &u_1 = 1,\ u_2 = \nu \ne 0,\ u_3 = 1 \\ &u_{n+3} = u_{n+2} + u_{n+1} + c \cdot u_n, \quad \text{for } n \ge 1. \end{aligned}$$

Furthermore, let W_1, W_2 denote the following subsystems of W:

$$W_1 = \{\mathbf{u} \in W : u_2 | u_4 \text{ and } f = -1\}$$

$$W_2 = \{\mathbf{u} \in W : u_2 | u_4 \text{ and } f \ne -1\}$$

where f is the integer from (1). Obviously, W_1 and W_2 are disjoint and

$$D \cap W \subseteq W_1 \cup W_2.$$

Proposition 3.1: The system of sequences W_1 contains no strong divisibility sequences, i.e., $W_1 \cap D = \emptyset$.

Proof: Let $\mathbf{u} \in W_1 \cap D$; then $b + f = 0$ and, according to Theorem 3.1 of [1], we get $\mathbf{u} = \mathbf{c}$ or $\mathbf{u} = \mathbf{d}$ where

$$\mathbf{c} = \{1, 2, 1, 0, 1, 2, 1, 0, \ldots\}, \quad \mathbf{d} = \{1, -2, 1, 0, 1, -2, 1, 0, \ldots\}.$$

But $\mathbf{c}, \mathbf{d} \notin W$ and thus $\mathbf{u} \notin W_1$, a contradiction.

Lemma 3.2: Let $\mathbf{u} = \{u_n\} \in W_2$. Then:

$$(5) \qquad c = f \cdot \nu - 1,$$

$$(6) \qquad u_4 = \nu \cdot (f + 1) \ne 0,$$

$$(7) \qquad c \equiv -\nu - 1 \pmod{|u_4|}.$$

Proof: The assertion (5) follows from (1), the assertions (6) and (7) follow from $u_4 = 1 + \nu + c$, from (5), and from the definition of W_2.

Lemma 3.3: Let $\mathbf{u} = \{u_n\} \in W_2 \cap D$, such that $f \ne 0$. Then $\nu \ne -1$.

Proof: Let us suppose that $\mathbf{u} \in W_2 \cap D$, $f \ne 0$, and $\nu = -1$. Then from (6) and (4) we get $0 \ne u_4 = c$ and consequently

$$u_{n+3} \equiv u_{n+2} + u_{n+1} \pmod{|u_4|}, \quad \text{for } n \ge 1.$$

Thus, $u_8 \equiv 3 \pmod{|u_4|}$ and from $u_4 | u_8$ we obtain $u_4 = c = \pm 1, \pm 3$. But

$c = 1 \Rightarrow \mathbf{u} \notin D$ (by Theorem 2.1), a contradiction
$c = -1 \Rightarrow f = 0$ [by (5)], a contradiction
$c = 3 \Rightarrow (u_9, u_{10}) \ne |u_1| \Rightarrow \mathbf{u} \notin D$, a contradiction
$c = -3 \Rightarrow (u_6, u_7) \ne |u_1| \Rightarrow \mathbf{u} \notin D$, a contradiction.

Lemma 3.4: Let $\mathbf{u} = \{u_n\} \in W_2$. Then $u_4 | u_8$ if and only if

$$\nu^2 \equiv \nu + 5 \pmod{|f + 1|}.$$

Proof: Using (7) and (4) we get $u_5 \equiv 1 - \nu - \nu^2 \pmod{|u_4|}$, then

(8) $\quad u_6 \equiv -\nu(\nu + 2) \pmod{|u_4|}, \quad u_7 \equiv -2\nu^2 - 3\nu + 1 \pmod{|u_4|}$

and, finally,

$$u_8 \equiv \nu(\nu^2 - \nu - 5) \pmod{|u_4|}.$$

But by (6), $u_4 = \nu \cdot (f + 1)$ and, therefore:

$$u_4 | u_8 \text{ if and only if } \nu^2 - \nu - 5 \equiv 0 \pmod{|f + 1|}.$$

Lemma 3.5: Let $\mathbf{u} = \{u_n\} \in W_2$ such that $u_4 | u_8$ and $u_4 | u_{12}$. Then

$$33\nu + 60 \equiv 0 \pmod{|f + 1|}.$$

Proof: From (7) and (6) we obtain $c \equiv -\nu - 1 \pmod{|f + 1|}$. Using this fact, (8), Lemma 3.4, (4), and the assumptions $u_4 | u_8$, $u_4 | u_{12}$, we get:

$$u_6 \equiv -3\nu - 5 \pmod{|f + 1|}, \qquad u_7 \equiv -5\nu - 9 \pmod{|f + 1|},$$
$$u_8 \equiv 0 \pmod{|f + 1|}, \qquad u_9 \equiv 6\nu + 11 \pmod{|f + 1|},$$
$$u_{10} \equiv 25\nu + 45 \pmod{|f + 1|}, \quad u_{11} \equiv 31\nu + 56 \pmod{|f + 1|},$$

and, finally,

$$u_{12} \equiv 33\nu + 60 \equiv 0 \pmod{|f + 1|}.$$

Proposition 3.6: Let $\mathbf{u} = \{u_n\} \in W_2$ such that $u_4 | u_8$ and $u_4 | u_{12}$. Then $f + 1 | 135$.

Proof: From Lemma 3.4, we get:

(9) $\quad 1089\nu^2 \equiv 1089\nu + 5445 \pmod{|f + 1|}.$

Similarly, from Lemma 3.5, we get:

(10) $\quad 1089\nu^2 \equiv 3600 \pmod{|f + 1|};$

(11) $\quad 1089\nu \equiv -1980 \pmod{|f + 1|}.$

Now, from (9), (10), and (11) we obtain

$$3600 \equiv 3465 \pmod{|f + 1|}$$

and thus, $f + 1 | 135$.

Lemma 3.7: Let $\mathbf{u} = \{u_n\} \in W_2$. Then $u_5 \neq 0$ and

(12) $\quad u_{10} \equiv \nu \cdot (f^3 - 5f^2 - 2f + 1) + f^2 - 4f - 6 \pmod{|u_5|}.$

Proof: From (5), (6), and (4) we get:

(13) $\quad u_5 = \nu^2 f + \nu f + 1.$

If $u_5 = 0$, then $\nu f \cdot (\nu + 1) = -1$ and thus, $\nu + 1 = \pm 1$, a contradiction. Furthermore, by a direct computation from (4), using (5), we get:

(14) $\quad u_{10} = \nu^3 f^3 + 6\nu^3 f^2 + 10\nu^2 f^2 + 6\nu^2 f + 10\nu f + \nu.$

From (13) we get $\nu^2 f \equiv -\nu f - 1 \pmod{|u_5|}$; using this fact in (14), we obtain (12).

Proposition 3.8: Let $\mathbf{u} = \{u_n\} \in W_2$ such that $u_5 | u_{10}$. Then

$$u_5 | f^4 - 13f^3 + 34f^2 + 38f + 1.$$

Proof: Let us denote $\alpha = f^3 - 5f^2 - 2f + 1$; $\beta = f^2 - 4f - 6$. Obviously,

$$\alpha^2 - \beta f(\alpha - \beta) = \alpha^2(\nu^2 f + \nu f + 1) - (\nu\alpha + \beta)(\alpha f\nu + f(\alpha - \beta)). \tag{15}$$

Then from $u_5 | u_{10}$, (12), (13), and (15), we obtain

$$u_5 | \alpha^2 - \beta f(\alpha - \beta) = f^4 - 13f^3 + 34f^2 + 38f + 1$$

which completes the proof of the proposition.

Now, let us denote by H the following subsystem of the system W:

$$H = \{\mathbf{u} \in W : c = -1\},$$

i.e., $\mathbf{u} \in H$ if and only if $\mathbf{u} = \{1, v, 1, v, \ldots\}$. It is obvious that $H \subseteq W_2$.

Proposition 3.9: Let $\mathbf{u} = \{u_n\} \in W_2$. Then $\mathbf{u} \in D$ if and only if $\mathbf{u} \in H$.

Proof: If $\mathbf{u} \in H$, then clearly $\mathbf{u} \in D$. Conversely, let $\mathbf{u} \in D$; then (by Proposition 3.6), $f + 1 | 135$. From Lemma 3.4 and from the fact that the congruence $\nu^2 \equiv \nu + 5 \pmod 9$ has no solution, we get $|f + 1| \neq 9, 27, 45, 135$. Therefore, we obtain for f the following eight possibilities: $f = 0, 2, 4, 14, -2, -4, -6, -16$. Now:

(i) let $f = 0$, then by (5), $c = -1$; thus, $\mathbf{u} = \{1, v, 1, v, \ldots\} \in H$.

(ii) let $f \neq 0$ and let us denote $\delta = f^4 - 13f^3 + 34f^2 + 38f + 1$. The possible values of f and the factorization of the corresponding δ are given in the table:

f	2	4	14	-2	-4	-6	-16
δ	5^3	11^2	9941	181	1481	5101	$181 \cdot 701$

But $u_5 | \delta$ (by Proposition 3.8), which gives us 38 possible pairs $\{f, u_5\}$. For a given pair $\{f, u_5\}$, we obtain the value ν from (13). Obviously, ν must be an integer and $\nu \neq 0, -1$ [by (4) and Lemma 3.3]. By a direct computation, we obtain the following solutions:

$$f = 2,\ \nu = 1, 3, -2, -4, \text{ and } f = 4,\ \nu = 5, -6.$$

For $f = 2$, $\nu = -4$, we get $(u_4, u_{11}) \neq |u_1|$; for $f = 4$, $\nu = 5$, we get $(u_5, u_6) \neq |u_1|$, and in the remaining cases we get $\nu^2 \not\equiv \nu + 5 \pmod{|f + 1|}$ and, therefore, by Lemma 3.4, $u_4 \nmid u_8$. Thus $\mathbf{u} \notin D$, a contradiction.

The following theorem gives a complete characterization of all the strong divisibility sequences in the system W.

Theorem 3.10: Let $\mathbf{u} \in W$. Then $\mathbf{u}$ is a strong divisibility sequence if and only if $\mathbf{u} \in H$.

Proof: The assertion follows immediately from Propositions 3.1 and 3.9 and from the inclusion $D \cap W \subseteq W_1 \cup W_2$.

Acknowledgment

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References

1. P. Horák. "A Note on the Third-Order Strong Divisibility Sequences." *Fibonacci Quarterly* 26.4 (1988):366-71.

2. P. Horák & L. Skula. "A Characterization of the Second-Order Strong Divisibility Sequences." *Fibonacci Quarterly 23.2* (1985):126-32.
3. C. Kimberling. "Strong Divisibility Sequences with Nonzero Initial Term." *Fibonacci Quarterly 16.6* (1978):541-44.

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At the request of Professor Lester Lange and with the permission of Professor Leonard Gillman, we have simply lifted Professor Gillman's delightful, melodic note, below, from page 375 of the June-July 1982 issue of *The American Mathematical Monthly*. Students need to know that the well-known limit mentioned involves the golden mean.

Gerald E. Bergum
Editor

MISCELLANEA

77.

Leonid Hambro, the well-known pianist, told me recently that he was about to enter a billiards tournament in which he would play 12 games; he knew the opposition, he said, and he estimated his odds for winning any particular game as 8 to 5. "What do you think your chances are of sweeping all 12 games?" I asked him. "They're pretty small," he said. "The probability that I'll win any one game is 8/13. To find the probability that I'll win all 12 you have to take 8/13 to the 12th power. That's a pretty small number."

He did not have a calculator in his pocket. But he had a pencil and a pad—and an inspiration. "Hey!" he said. "Those are Fibonacci numbers. The ratio of successive terms approaches a limit (about .618), and very fast: even a ratio near the beginning like 8/13 is very close to the limit." He scribbled some additions. "The 12th Fibonacci number after 8 is 2584. Therefore 8/13 to the 12th power is approximately the same as 8/13 times 13/21 and so on, twelve times; everything cancels out except the 8 in the beginning and the 2584 at the end. So the probability that I will win all 12 games is about 8/2584, or about 1/300. See, I told you it was pretty small."

—Leonard Gillman
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