ON A GENERALIZATION OF A RECURSIVE SEQUENCE

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1. Introduction

Let k and t be fixed positive integers and let $G_{k,t}(n)$, $n = 0, 1, 2, \ldots$, be a sequence of integers defined by

$$(1)\qquad G_{k,t}(n) = \begin{cases} n & \text{if } 0 \le n \le t-1 \\ n - G_{k,t}^{k}(n-t) & \text{if } n \ge t, \end{cases}$$

where $G_{k,t}^{k}$ denotes the k^{th} iterated composition of $G_{k,t}$, i.e.,

$$G_{k,t}^{1}(m) = G_{k,t}(m) \quad \text{and} \quad G_{k,t}^{i}(m) = G_{k,t}(G_{k,t}^{i-1}(m))$$

for $i > 1$ and for any $m \ge 0$.

This sequence is a generalization of some which have been investigated earlier. P. J. Downey & R. E. Griswold [1] (and later V. Granville & J. P. Rasson [3]) proved that the solution of recurrence (1) in the case $k = 2$, $t = 1$ is given by

$$(2)\qquad G_{2,1}(n) = [(n+1)\mu]$$

for any $n \ge 0$, where $\mu = (-1 + \sqrt{5})/2$ and [ ] denotes the integer part function. In [1] a similar formula is shown for $G_{2,t}(n)$ with arbitrary $t \ge 1$.

Recently B. Zay [6] has shown some properties of the general sequence for any k and t. Among others he proved that $G_{k,t}(n)$ is defined for each nonnegative integer n, the sequence is monotonically increasing, and that the general case can be traced back to the case $t = 1$ by

$$G_{k,t}(n) = \begin{cases} t \cdot G_{k,1}\left(\left[\frac{n}{t}\right]\right) & \text{if } G_{k,1}\left(\left[\frac{n}{t}\right]\right) = G_{k,1}\left(\left[\frac{n}{t} + 1\right]\right) \\ t \cdot G_{k,1}\left(\left[\frac{n}{t}\right]\right) + n - t \cdot \left[\frac{n}{t}\right] & \text{if } G_{k,1}\left(\left[\frac{n}{t}\right]\right) \ne G_{k,t}\left(\left[\frac{n}{t} + 1\right]\right) \end{cases}$$

for any $n \ge 0$. So it is enough to investigate the sequence with $t = 1$. Furthermore, we can suppose that $k \ge 2$ since the case $k = t = 1$ gives the sequence $G_{1,1}(n) = [(n+1)/2]$, which can be considered as a trivial case.

Throughout this paper, k will denote a fixed integer with $k \ge 2$ and, for brevity, we write $G(n)$ instead of $G_{k,1}(n)$.

In general (if $k > 2$) the terms of the sequence $G(n)$ cannot be expressed similarly as in (2). In order to see it, let us suppose that there is an integer r and a positive real number ω such that

$$(3)\qquad G(n) = [(n+r)\omega].$$

Then

$$(4)\qquad \lim_{n\to\infty} \frac{G(n)}{n} = \omega.$$

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On the other hand, by (1) we have

$$\frac{G(n)}{n} = 1 - \frac{G^k(n-1)}{G^{k-1}(n-1)} \cdot \frac{G^{k-1}(n-1)}{G^{k-2}(n-1)} \cdots \frac{G^2(n-1)}{G(n-1)} \cdot \frac{G(n-1)}{n-1} \cdot \frac{n-1}{n};$$

therefore, $G^i(n) = G(G^{i-1}(n))$ and (4) imply the equation

$$\omega = 1 - \omega^k.$$

So ω is the only positive real root of the equation $x^k + x - 1 = 0$. But it can be checked by numerical calculation that, in the case $k = 3$, equation (3), with any integer r, does not hold for all n. Namely, in this case, we have $\omega = 0.6823\ldots$, $G(2) = 1$, $G(18) = 13$; thus, from

$$G(2) = 1 = [(2 + r)\omega] \quad \text{and} \quad G(18) = 13 = [(18 + r)\omega],$$

$r < 1$ and $r > 1$ would follow, respectively, which is impossible.

Thus, (2) really cannot be extended for any $k \geq 2$. But we shall show that (4) holds for any k.

Theorem: For any integer $k \geq 2$,

$$\lim_{n \to \infty} \frac{G(n)}{n} = \omega,$$

where ω is the single positive real root of the equation $x^k + x - 1 = 0$.

We note that the Theorem also holds if $t > 1$ or $k = 1$, which follows from the results mentioned above.

2. Auxiliary Results

For the proof of our Theorem, we need the following lemmas.

Lemma 1: For any $n > 0$, we have

(5) $$G(n) = G(n-1) + \varepsilon_n$$

and

(6) $$G^k(n) = G^k(n-1) + \varepsilon_n',$$

where ε_n and ε_n' are 0 or 1.

Proof: Equalities (5) and (6) hold for $n = 1$ and $n = 2$ since, by the definition of the sequence,

$$G(0) = 0,\ G^k(0) = 0,\ G(1) = 1,\ G(2) = 1,\ G^k(1) = 1,\ G^k(2) = 1$$

for any $k \geq 2$. Assume that $m \geq 2$ and (5) holds for any $n \leq m$, i.e.,

$$G(n) = G(n-1) + \varepsilon_n$$

for any n with $0 < n \leq m$ and $\varepsilon_n = 0$ or 1. From this $G(n) \leq n \leq m$ also follows and so, by the assumption, we get

$$G(G(n)) = G^2(n) = \begin{cases} G^2(n-1) & \text{if } \varepsilon_n = 0 \\ G^2(n-1) + \varepsilon_n'' & \text{if } \varepsilon_n = 1, \end{cases}$$

where $\varepsilon_n'' = 0$ or 1. Continuing this process,

(7) $$G^k(n) = G^k(n-1) + \varepsilon_n' \quad (\varepsilon_n' = 0 \text{ or } 1)$$

follows for any $0 < n \leq m$. By (1) we have

$$G(m) = m - G^k(m-1) \quad \text{and} \quad G(m+1) = m + 1 - G^k(m)$$

from which, using (7), we obtain

$$G(m+1) - G(m) = 1 - (G^k(m) - G^k(m-1)) = \varepsilon_{m+1} \quad (\varepsilon_{m+1} = 0 \text{ or } 1).$$

Thus, (5), (7), and (6) also hold for $n = m + 1$.

From these, the lemma follows by mathematical induction.

Lemma 2: Let $\{n_i\}_{i=0}^{\infty}$ be a sequence of positive integers such that

$$G(n_i) = n_{i-1}$$

for any $i > 0$. Then

$$n_i = n_{i-1} + n_{i-k} - \varepsilon_i$$

for any $i \geq k$, where $\varepsilon_i = 0$ or 1.

Proof: By the assumption of the lemma, using Lemma 1 and the definition of the sequence $G(n)$, for any $i \geq k$ we have

$$\begin{aligned} n_{i-1} = G(n_i) &= n_i - G^k(n_i - 1) = n_i - G^k(n_i) + \varepsilon_i' \\ &= n_i - G^{k-1}(n_{i-1}) + \varepsilon_i' = n_i - G^{k-2}(n_{i-2}) + \varepsilon_i' = \cdots \\ &= n_i - G(n_{i-k+1}) + \varepsilon_i' = n_i - n_{i-k} + \varepsilon_i', \end{aligned}$$

where $\varepsilon_i' = 0$ or 1. The lemma follows from this assertion.

Lemma 3: Let $\{n_i\}_{i=0}^{\infty}$ be an increasing sequence of nonnegative integers satisfying the recursion

$$n_i = n_{i-1} + n_{i-k} - \varepsilon_i \quad (i \geq k),$$

where $k \geq 2$ is a fixed positive integer and $\varepsilon_i = 0$ or 1. Define a k^{th}-order linear recurrence sequence $\{u_i\}_{i=0}^{\infty}$ of integers by $u_i = n_i$ for $0 \leq i \leq k - 1$ and

$$u_i = u_{i-1} + u_{i-k}$$

for $i \geq k$. Further, let $\{F_i\}_{i=0}^{\infty}$ be a sequence of natural numbers defined by $F_0 = F_1 = \cdots = F_{k-1} = 1$ and

$$F_i = F_{i-1} + F_{i-k} \quad (i \geq k).$$

Then

$$n_i = u_i - \delta_i$$

for any $i \geq 0$, where $0 \leq \delta_i \leq F_i - 1$.

Proof: For $0 \leq i \leq k - 1$, the lemma evidently holds with $\delta_i = 0$. If $i \geq k$ and $n_j = u_j - \delta_j$ with $0 \leq \delta_j \leq F_j - 1$ for any $0 \leq j < i$, then

$$\begin{aligned} n_i &= n_{i-1} + n_{i-k} - \varepsilon_i \\ &= u_{i-1} + u_{i-k} - (\delta_{i-1} + \delta_{i-k} + \varepsilon_i) = u_i - \delta_i, \end{aligned}$$

where

$$0 \leq \delta_i = \delta_{i-1} + \delta_{i-k} + \varepsilon_i \leq F_{i-1} + F_{i-k} - 2 + \varepsilon_i \leq F_i - 1,$$

since the δ_j's are integers. The lemma follows from the above by mathematical induction on i.

Lemma 4: Let $\{v_n\}_{n=0}^{\infty}$ be a k^{th}-order linear recurrence sequence of positive rational integers defined by the nonzero initial values $v_0, v_1, \ldots, v_{k-1}$ and by the recursion

$$v_n = v_{n-1} + v_{n-k}$$

for $n \geq k$. Denote by $\alpha_1, \alpha_2, \ldots, \alpha_k$ the roots of the characteristic polynomial $x^k - x^{k-1} - 1$. Then the terms of the sequence can be expressed as

$$v_n = a_1\alpha_1^n + a_2\alpha_2^n + \cdots + a_k\alpha_k^n \quad (n \geq 0), \tag{8}$$

where the a_i's ($i = 1, 2, \ldots, k$) are elements of the number field generated by $\alpha_1, \alpha_2, \ldots, \alpha_k$ over the rationals.

Proof: This lemma is a special case of a more general well-known result, so it is not necessary to prove it here.

Lemma 5: Let $\{v_n\}_{n=0}^{\infty}$ be the linear recurrence sequence defined in Lemma 4. If

$$0 < v_0 = \min_{0 \le i < k} (v_i) \quad \text{and} \quad |\alpha_1| > |\alpha_i| \text{ for } 2 \le i \le k,$$

then there is a real number $c > 0$, depending only on the characteristic polynomial of the sequence, such that

$$|a_1| > c \cdot v_0, \tag{9}$$

where a_1 is defined by (8).

Proof: Ferguson [2] as well as Hoggatt & Alladi [4] proved that the roots of the polynomial $x^k - x^{k-1} - 1$ are distinct and that there is a dominant real root α_1 with the largest modulus; thus, we may suppose that $|\alpha_1| > |\alpha_i|$ for $i = 2, \ldots, k$.

By (8), for the a_i's, we have the system equations:

$$\begin{aligned}
a_1 \quad + a_2 \quad + \cdots + a_k \quad &= v_0\\
a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k &= v_1\\
&\vdots\\
a_1\alpha_1^{k-1} + a_2\alpha_2^{k-1} + \cdots + a_k\alpha_k^{k-1} &= v_{k-1};
\end{aligned}$$

thus,

$$a_1 = \frac{D_1}{D}, \tag{10}$$

where

$$D = \begin{vmatrix} 1 & 1 & \ldots & 1\\ \alpha_1 & \alpha_2 & \ldots & \alpha_k\\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_k^2\\ \vdots & & & \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \ldots & \alpha_k^{k-1} \end{vmatrix}, \quad D_1 = \begin{vmatrix} v_0 & 1 & \ldots & 1\\ v_1 & \alpha_2 & \ldots & \alpha_k\\ v_2 & \alpha_2^2 & \ldots & \alpha_k^2\\ \vdots & & & \\ v_{k-1} & \alpha_2^{k-1} & \ldots & \alpha_k^{k-1} \end{vmatrix},$$

and $D \neq 0$ since the α_i's are distinct. The determinant D_1 can be written in the form

$$D_1 = \sum_{i=1}^{k} (-1)^{i-1} v_{i-1} \cdot D^{(i)}, \tag{11}$$

where

$$D^{(i)} = \begin{vmatrix} 1 & \ldots & 1\\ \alpha_2 & \ldots & \alpha_k\\ \vdots & & \\ \alpha_2^{i-2} & \ldots & \alpha_k^{i-2}\\ \alpha_2^{i} & \ldots & \alpha_k^{i}\\ \vdots & & \\ \alpha_2^{k-1} & \ldots & \alpha_k^{k-1} \end{vmatrix}$$

is a $(k - 1) \times (k - 1)$ determinant rejecting the first column and the i^{th} row from D_1.

It was proved in the lemma of [5] that

$$D^{(i)} = D_0 \cdot S_{k-i} \text{ for any } 1 \le i \le k, \tag{12}$$

where

$$D_0 = \begin{vmatrix} 1 & \dots & 1 \\ \alpha_2 & \dots & \alpha_k \\ \vdots & & \\ \alpha_2^{k-2} & \dots & \alpha_k^{k-2} \end{vmatrix}$$

is a $(k - 1) \times (k - 1)$ Vandermonde determinant and S_{k-i} is the elementary symmetrical polynomial of degree $k - i$ of variables $\alpha_2, \dots, \alpha_k$ if $k - i > 0$, and $S_0 = 1$. It is known that for the coefficients of a polynomial

$$b(x) = b_0x^n + b_1x^{n-1} + \dots + b_n$$

we have

$$b_j = (-1)^j b_0 S_j' \quad (1 \le j \le n)$$

where

$$S_j' = \sum \beta_{i_1}\beta_{i_2} \dots \beta_{i_j}$$

is the elementary symmetrical polynomial of degree j of the roots $\beta_1, \dots, \beta_n$ of $b(x)$ (the sum runs over the distinct $i_1 < i_2 < \dots < i_j$ combinations of 1, 2, ..., n). Since $S_1, S_2, \dots, S_{k-1}$ are the elementary symmetrical polynomials of $\alpha_2, \dots, \alpha_k$ of degree 1, 2, ..., $k - 1$, thus $S_1 + \alpha_1$, $S_2 + S_1\alpha_1$, ..., $S_{k-1} + S_{k-2}\alpha_1$, $S_{k-1}\alpha_1$ are the elementary symmetrical polynomials of $\alpha_1, \alpha_2, \dots, \alpha_k$ of degree 1, 2, ..., $k - 1$, k, respectively. So, for the coefficients of the polynomial $x^k - x^{k-1} - 1$, we have

$$\begin{aligned} -1 &= -(S_1 + \alpha_1) \\ 0 &= S_2 + S_1\alpha_1 \\ &\vdots \\ 0 &= (-1)^{k-1}(S_{k-1} + S_{k-2}\alpha_1) \\ -1 &= (-1)^k \cdot S_{k-1}\alpha_1. \end{aligned} \tag{13}$$

Since α_1 is real, $\alpha_1 > 1$, which implies that $S_1 = 1 - \alpha_1 > 0$. But, from this, $S_2 > 0$ follows, and contiuing this process, by (13), we obtain the inequalities

$$S_{2i} > 0 \quad (0 \le 2i \le k - 1) \tag{14}$$

and

$$S_{2i+1} < 0 \quad (1 \le 2i + 1 \le k - 1). \tag{15}$$

Finally, by (11) and (12) we get

$$D_1 = D_0(v_0S_{k-1} - v_1S_{k-2} + \dots + (-1)^{k-1}v_{k-1}S_0)$$

and, by (14) and (15), using the condition $0 < v_0 \le v_i$ for $1 \le i \le k - 1$,

$$|D_1| = |D_0| \cdot \sum_{i=1}^{k} v_{i-1} \cdot |S_{k-i}| > v_0 \cdot |D_0| \cdot \sum_{i=1}^{k} |S_{k-i}|$$

follows. By (10), this implies the lemma.

3. Proof of the Theorem

Let N be a sufficiently large positive integer and define an integer m by

$$m = \left[\frac{\log N}{2 \cdot \log 3}\right]$$

([ ] is the integer part function). Let $n_0, n_1, \dots, n_m$ be a set of natural numbers defined by

$$n_m = N \quad \text{and} \quad n_{i-1} = G(n_i) \quad \text{for } 1 \le i \le m. \tag{16}$$

From Lemma 1 and its proof, it follows that $G(n) < n$ for any $n > 1$, and so

$$n_0 < n_1 < \cdots < n_m = N$$

for N sufficiently large so that $n_0 \geq 1$.

We show that there are no three consecutive equal terms in the sequence $G(n)$. For if

$$G(n) = G(n + 1) = G(n + 2),$$

then, by the definition of the sequence,

$$(17) \qquad n - G^k(n - 1) = n + 1 - G^k(n) = n + 2 - G^k(n + 1)$$

would follow. But $G(n) = G(n + 1)$ implies that $G^k(n) = G^k(n + 1)$ and so, by (17), we would obtain the equality $n + 1 = n + 2$, which is impossible. Thus, $G(n + 2) \geq G(n) + 1$ for any $n \geq 0$, and so

$$(18) \qquad G(n) \geq \frac{1}{3}n.$$

By (16) and (18), we get

$$N = n_m \leq 3 \cdot G(n_m) = 3 \cdot n_{m-1} \leq 3^2 \cdot G(n_{m-1}) = 3^2 \cdot n_{m-2} \leq \cdots \leq 3^m n_0,$$

which, by the definition of m, can be written in the form

$$(19) \qquad n_0 \geq \frac{N}{3^m} \geq \sqrt{N}.$$

By Lemmas 2-4 and their notations, using (16), we obtain

$$(20) \qquad \frac{G(N)}{N} = \frac{n_{m-1}}{n_m} = \frac{u_{m-1} - \delta_{m-1}}{u_m - \delta_m} = \frac{a_1\alpha_1^{m-1} + \cdots + a_k\alpha_k^{m-1} - \delta_{m-1}}{a_1\alpha_1^m + \cdots + a_k\alpha_k^m - \delta_m}$$

$$= \frac{1}{\alpha_1} \cdot \frac{1 + \dfrac{a_2}{a_1}\left(\dfrac{\alpha_2}{\alpha_1}\right)^{m-1} + \cdots + \dfrac{a_k}{a_1}\left(\dfrac{\alpha_k}{\alpha_1}\right)^{m-1} - \dfrac{1}{a_1} \quad \delta_{m-1}/\alpha_1^{m-1}}{1 + \dfrac{a_2}{a_1}\left(\dfrac{\alpha_2}{\alpha_1}\right)^{m} + \cdots + \dfrac{a_k}{a_1}\left(\dfrac{\alpha_k}{\alpha_1}\right)^{m} - \dfrac{1}{a_1} \cdot \delta_m/\alpha_1^m}.$$

By the proof of Lemma 5, it follows that there are complex numbers $b_1, b_2, \ldots, b_k$, which depend only on the α_i's ($i = 1, 2, \ldots, k$), such that

$$a_i = \sum_{i=0}^{k-1} b_i u_i$$

and so, using that $|a_1| > c \cdot u_0$ by Lemma 5,

$$(21) \qquad \left|\frac{a_i}{a_1}\right| < \frac{\left|\sum_{i=0}^{k-1} b_i u_i\right|}{c \cdot u_0}$$

follows. But $u_i = n_i$ for $i = 0, 1, 2, \ldots, k - 1$, $n_i < n_{k-1}$ for $0 \leq i < k - 1$, and by (18) $n_i/n_{i-1} \leq 3$ for any $i > 0$; thus, from (21),

$$(22) \qquad \left|\frac{a_i}{a_1}\right| < b \cdot \frac{n_{k-1}}{n_0} = b \cdot \frac{n_1}{n_0} \cdot \frac{n_2}{n_1} \cdot \cdots \cdot \frac{n_{k-1}}{n_{k-2}} \leq b \cdot 3^{k-1} = B$$

follows for $2 \leq i \leq k$, where b and B are positive real numbers which do not depend on m and the n_i's. Since $|\alpha_1| > |\alpha_i|$ for $2 \leq i \leq k$, and $m \to \infty$ as $N \to \infty$, so by (22),

(23) $$\lim_{N\to\infty} \frac{a_i}{a_1}\left(\frac{\alpha_i}{\alpha_1}\right)^{m-1} = \lim_{N\to\infty} \frac{a_i}{a_1}\left(\frac{\alpha_i}{\alpha_1}\right)^{m} = 0 \quad \text{for } i = 2, 3, \ldots, k.$$

On the other hand, by Lemmas 3 and 4, we get

$$0 \le \delta_n < F_n = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_k\alpha_k^n = c_1\alpha_1^n\left(1 + \sum_{i=2}^{k} \frac{c_i}{c_1}\left(\frac{\alpha_i}{\alpha_1}\right)^n\right)$$

for any $n \ge 0$, where the c_i's ($i = 1, 2, \ldots, k$) are complex numbers which are independent of n,

$$\lim_{n\to\infty}(\alpha_i/\alpha_1)^n = 0,$$

and it can be easily seen that $c_1 \ne 0$. From these, it follows that there is a real number $C > 0$, depending only on the characteristic polynomial of the sequence $\{F_i\}$, such that

$$\left|\frac{\delta_n}{\alpha_1^n}\right| < C \text{ for any } n \ge 0.$$

However, by (19) and Lemma 5,

$$|a_1| > c \cdot u_0 = c \cdot n_0 \ge c \cdot \sqrt{N}$$

and so

(24) $$\lim_{N\to\infty}\left(\frac{1}{a_1} \cdot \frac{\delta_{m-1}}{\alpha_1^{m-1}}\right) = \lim_{N\to\infty}\left(\frac{1}{a_1} \cdot \frac{\delta_m}{\alpha_1^m}\right) = 0.$$

From (20), (23), and (24),

$$\lim_{N\to\infty} \frac{G(N)}{N} = \frac{1}{\alpha_1}$$

follows, where α_1 is the single positive root of the equation $x^k - x^{k-1} - 1 = 0$. But, if α is a root of the polynomial $x^k - x^{k-1} - 1$, then $1/\alpha$ is a root of $x^k + x - 1$, thus $1/\alpha_1 = \omega$ and the theorem is proved.

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