CONTINUED FRACTIONS AND PYTHAGOREAN TRIPLES

William C. Waterhouse*
The Pennsylvania State University, University Park, PA 16802
(Submitted June 1990)

Introduction

A Pythagorean triple is an ordered triple of positive integers (x, y, z) with $x^2 + y^2 = z^2$. It is called primitive if x and y have no common factors. In recent work, A. G. Schaake & J. C. Turner have discovered an unexpected representation for the primitive Pythagorean triples: they are precisely the triples of the form

$$x = (Q - R)/N, \quad y = (P + S)/N, \quad z = (Q + R)/N$$

where P/Q is the value of a continued fraction of the form

$$[0; u_1, u_2, \ldots, u_i, v, 1, j, (v + 1), u_i, \ldots, u_2, u_1],$$

R/S is the previous convergent of that continued fraction, and N depends on the entries but is either $(j + 1)$ or $2(j + 1)$. This work was drawn to my attention by the review [4]; Professor Turner was then kind enough to send me relevant parts of their privately published book and research report [2], [3]. The representation is derived there as part of a more general investigation, with the equation rewritten in the form $ps = qr - 1$. In this paper, I shall isolate the material bearing directly on Pythagorean triples, proving a slightly simpler variant of their result and showing how closely it is related to the usual parametrization of Pythagorean triples. The only unfamiliar step will be an identity on continued fractions that we can easily prove from scratch.

1. An Identity on Continued Fractions

We briefly recall some of the basic information about continued fractions (see, e.g., [1], Ch. IV). For positive integers $u_1, \ldots, u_m$, the continued fraction $[0; u_1, \ldots, u_m]$ is a number between 0 and 1 defined inductively by $[0; u] = 1/u$ and

$$[0; u_1, \ldots, u_m] = 1/\{u_1 + [0; u_2, \ldots, u_m]\}.$$

If we define two sequences p_j and q_j by the initial values

$$p_0 = 0, \; q_0 = 1, \; p_1 = 1, \; q_1 = u_1$$

and the recursion relations

$$p_{j+1} = u_{j+1}p_j + p_{j-1}, \; q_{j+1} = u_{j+1}q_j + q_{j-1},$$

then p_j and q_j are relatively prime and

$$[0; u_1, \ldots, u_m] = p_m/q_m.$$

Every fraction between 0 and 1 occurs as some $[0; u_1, \ldots, u_m]$, and the expression is unique so long as we require the last entry u_m to be bigger than 1.

Lemma: Let $[0, \sim\sim\sim] = A/B$ be a continued fraction, and let $[0; \sim\sim\sim, g] = C/D$. For any u, then

*This work was supported in part by the U.S. National Science Foundation Grant no. DMS8701690.

$$[0;\ u,\ \sim\sim\sim,\ g] = D/(uD + C)$$

and

$$[0;\ u,\ \sim\sim\sim,\ g,\ u] = (uD + B)/(u^2D + uB + uC + A).$$

Proof: Clearly,

$$D/(uD + C) = 1/\{u + [0;\ \sim\sim\sim,\ g]\}.$$

Similarly,

$$B/(uB + A) = [0;\ u,\ \sim\sim\sim].$$

Then the recursion relations show us that the numerator of $[0;\ u,\ \sim\sim\sim,\ g,\ u]$ is $uD + B$ and the denominator is $u(UD + C) + (uB + A)$. □

Theorem 1 (Schaake & Turner): Let $[0;\ u_n,\ u_{n-1},\ \ldots,\ u_1,\ w]$ be a continued fraction with the u_i and w positive integers, $w > 1$, and $n \geq 0$. Let its value be p/q. Then the continued fraction

$$[0;\ u_n,\ \ldots,\ u_2,\ u_1,\ w - 1,\ w + 1,\ u_1,\ u_2,\ \ldots,\ u_n]$$

has numerator $pq + (-1)^n$ and denominator q^2, and the previous convergent

$$[0;\ u_n,\ \ldots,\ u_2,\ u_1,\ w - 1,\ w + 1,\ u_1,\ u_2,\ \ldots,\ u_{n-1}]$$

has numerator p^2 and denominator $pq - (-1)^n$.

Proof: We prove this by induction on n (which is why the entries in the continued fraction have been numbered backward). The case $n = 0$ is straightforward: we have $[0;\ w] = 1/w$ and $[0;\ w - 1] = 1/(w - 1)$, while $[0;\ w - 1,\ w + 1]$ has numerator $(w + 1)$ and denominator $(w + 1)(w - 1) + 1 = w^2$.

Assuming the result for n, let us consider the fractions for $n + 1$. If p/q is the value for n, the lemma shows that

$$[0;\ u_{n+1},\ u_n,\ \ldots,\ u_2,\ u_1,\ w] = q/\{qu_{n+1} + p\}.$$

In short, the new numerator p' is q, and the new denominator q' is $qu_{n+1} + p$. Applied to the longer fractions, the lemma now shows that

$$[0;\ u_{n+1},\ \ldots,\ u_2,\ u_1,\ w - 1,\ w + 1,\ u_1,\ u_2,\ \ldots,\ u_n]$$
$$= q^2/\{u_{n+1}q^2 + pq + (-1)^n\} = (p')^2/\{p'q' - (-1)^{n+1}\},$$

while

$$[0;\ u_{n+1},\ \ldots,\ u_2,\ u_1,\ w - 1,\ w + 1,\ u_1,\ u_2,\ \ldots,\ u_{n+1}]$$

has numerator equal to

$$u_{n+1}(q^2) + \{pq - (-1)^n\} = p'q' + (-1)^{n+1}$$

and denominator equal to

$$u_{n+1}\{q^2u_{n+1} + pq + (-1)^n\} + \{(pq - (-1)\ )u_{n+1} + p^2\} = (q')^2. \quad \square$$

Example: Suppose we start with $[0;\ 3,\ 5,\ 2]$. Computing the sequence of values $(p_j,\ q_j)$ by the recursion, we get $(0,\ 1)$, $(1,\ 3)$, $(5,\ 16)$, and $(11,\ 35)$. For $[0;\ 3,\ 5,\ 1,\ 3,\ 5,\ 3]$, the sequence is $(0,\ 1)$, $(1,\ 3)$, $(5,\ 16)$, $(6,\ 19)$, $(23,\ 73)$, $(121,\ 384)$, and $(386,\ 1225)$. We see, e.g., that $386 = (11)(35) + (-1)^2$ and $1225 = (35)^2$.

Remark 1: I have stated the theorem for positive integers, but the proof shows that it is a purely formal identity.

Remark 2: In Schaake & Turner, Theorem 1 occurs as a special case (the formal result of setting $j = 0$) in a more general statement ([3], pp. 92-96); in our notation, it says that the continued fraction

$$[0;\ u_n,\ \ldots,\ u_2,\ u_1,\ (w - 1),\ 1,\ j,\ w,\ u_1,\ u_2,\ \ldots,\ u_n]$$

has numerator $(j + 1)pq + (-1)^n$ and denominator $(j + 1)q^2$, while the previous convergent has numerator $(j + 1)p^2$ and denominator $(j + 1)pq - (-1)^n$. The induction argument given here will also establish that statement. Their proof is slightly different, and in [2] the result is viewed primarily as a computational simplification. From my present viewpoint, the more general expression simply winds up introducing a common factor of $(j + 1)$ in the Pythagorean triples, and so it is not needed.

2. Relation to Pythagorean Triples

Now we recall the standard analysis of Pythagorean triples, as in ([1], pp. 153-55). If (x, y, z) is a Pythagorean triple, then so is (mx, my, mz) for any positive integer m. Every Pythagorean triple arises in this way from a uniquely determined primitive triple. Setting $\bar{x} = x/z$ and $\bar{y} = y/z$, we thus get a correspondence between the primitive triples and the points with rational coordinates on the first quadrant of the unit circle. What we might call the "standard" rational parameter for the circle is

$$t = y/(x + z) = \bar{y}/(\bar{x} + 1);$$

the values $\bar{x}$ and $\bar{y}$ (with squares adding to 1) can be recovered as

$$\bar{x} = (1 - t^2)/(1 + t^2) \quad \text{and} \quad \bar{y} = 2t/(1 + t^2).$$

We get positive values for $\bar{x}$ and $\bar{y}$ exactly when $0 < t < 1$. If $t = p/q$ in lowest terms, we have

$$\bar{x} = (p^2 - q^2)/(p^2 + q^2) \quad \text{and} \quad \bar{y} = 2pq/(q^2 + p^2).$$

The obvious Pythagorean triple corresponding to this value of t then is

$$x = p^2 - q^2, \quad y = 2pq, \quad z = p^2 + q^2.$$

This triple is in fact the primitive one if either p or q is even. If both of them are odd, then the primitive triple for parameter $t = p/q$ is

$$x = (p^2 - q^2)/2, \quad y = pq, \quad z = (p^2 + q^2)/2.$$

The classification is sometimes stated a bit differently, so I should add one further remark. Interchanging x and y in a Pythagorean triple gives another Pythagorean triple. On the rational parameter, this corresponds to the operation sending t to $(1 - t)/(1 + t)$. If $t = p/q$ in lowest terms, the new value is $(q - p)/(q + p)$. This new numerator and denominator have at most a common factor of 2. If either p or q is even, then the fraction is in lowest terms and has odd numerator and denominator. If both p and q are odd, then either $p - q$ or $p + q$ (but not both) is divisible by 4; hence, when we cancel the common factor of 2, we get a fraction where either the numerator or the denominator is even. Thus, the two possible types of t are interchanged by the interchange of x and y. Specifically, the p/q with either p or q even give the primitive triples in which y is even, while the p/q with both p and q odd give the primitive triples where x is even.

We can now prove the main result, showing how the continued fraction is related to the rational parameter.

Theorem 2:

(a) For $n \geq 0$ and any positive integers $w, u_1, u_2, \ldots, u_n$ with $w > 1$, let P/Q be the value of the continued fraction

$$[0; u_n, \ldots, u_2, u_1, w - 1, w + 1, u_1, u_2, \ldots, u_n],$$

and let R/S be the value of its previous convergent

$$[0; u_n, \ldots, u_2, u_1, w - 1, w + 1, u_1, u_2, \ldots, u_{n-1}].$$

Set $N = 2$ if both Q and R are odd, and set $N = 1$ otherwise. Define

$$X = (Q - R)/N, \quad Y = (P + S)/N, \quad Z = (Q + R)/N.$$

Then (X, Y, Z) is a primitive Pythagorean triple.

(b) Every primitive Pythagorean triple arises in this way from exactly one sequence $w, u_1, u_2, \ldots, u_n$.

(c) The rational parameter for the triple is precisely $[0; u_n, u_{n-1}, \ldots, u_1, w]$.

Proof: Set $t = p/q = [0; u_n, \ldots, u_2, u_1, w]$. By Theorem 1, we know that

$$P = pq + (-1)^n, \quad Q = q^2, \quad R = p^2, \quad S = pq - (-1)^n.$$

Thus, $Q - R = q^2 - p^2$ and $Q + R = q^2 + p^2$ and $P + S = 2pq$. The standard theory shows then that (X, Y, Z) is the primitive triple corresponding to parameter t. As each t arises from a unique sequence, the same is true for the triples. □

References

1. H. Davenport. *The Higher Arithmetic*. New York: Dover, 1983 (a reprint of the original edition [London: Hutchison], 1952).
2. A. G. Schaake & J. C. Turner. *A New Chapter for Pythagorean Triples*. Privately published by the authors, Hamilton, New Zealand, 1989.
3. A. G. Schaake & J. C. Turner. "New Methods for Solving Quadratic Diophantine Equations." Research Report 192, University of Waikato, Hamilton, New Zealand, 1989.
4. Review of *A New Chapter for Pythagorean Triples* by A. G. Schaake & J. C. Turner. *Fibonacci Quarterly* *28.2* (1990):140 and 155.

*****