ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

<u>H-469</u> *Proposed by H.-J. Seiffert, Berlin, Germany*

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ \text{for } n \ge 2.$$

Show that for all positive integers n and all positive reals x,

(a) $$\frac{1}{F_{2n-1}(x)} = \frac{x^2+4}{2n-1}\sum_{k=0}^{2n-2}(-1)^{k+n+1}\frac{\cos\frac{k\pi}{2n-1}}{x^2+4\cos^2\frac{k\pi}{2n-1}},$$

(b) $$\frac{1}{F_{2n}(x)} = \frac{x(x^2+4)}{4n}\sum_{k=0}^{2n-1}\frac{(-1)^{k+n}}{x^2+4\cos^2\frac{k\pi}{2n}}.$$

<u>H-470</u> *Proposed by Paul S. Bruckman, Edmonds, WA*

Consider the polynomial

(1) $$G_r(z) = z^r - \sum_{k=0}^{r-1} a_k z^{r-1-k},\ r \ge 1, \text{ the } a_k\text{'s complex.}$$

Consider the r distinct sequences $(U_{n,j}^{(r)})_{n=0}^{\infty}$ satisfying the common recurrence relation:

(2) $$G_r(E)(U_{n,j}^{(r)}) = 0,\ j = 1, 2, \ldots, r;\ n = 0, 1, \ldots .$$

The sequences are specified by the initial values:

(3) $$U_{n,j}^{(r)} = \delta_{n+j,r},\ n = 0, 1, \ldots, r-1,\ j = 1, 2, \ldots, r.$$

Form the $r \times r$ matrix $U_n^{(r)}$, defined as follows:

(4) $$U_n^{(r)} = \begin{bmatrix} U_{n+r-1,1}^{(r)} & U_{n+r-1,2}^{(r)} & \cdots & U_{n+r-1,r}^{(r)} \\ U_{n+r-2,1}^{(r)} & U_{n+r-2,2}^{(r)} & \cdots & U_{n+r-2,r}^{(r)} \\ \vdots & \vdots & & \vdots \\ U_{n+1,1}^{(r)} & U_{n+1,2}^{(r)} & \cdots & U_{n+1,r}^{(r)} \\ U_{n,1}^{(r)} & U_{n,2}^{(r)} & \cdots & U_{n,r}^{(r)} \end{bmatrix} = ((U_{n+r-i,j}^{(r)})).$$

Therefore,

$$(5) \qquad U_1^{(r)} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{r-2} & a_{r-1} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

(A) Find the characteristic polynomial $p_r(z)$ of $U_1^{(r)}$;

(B) Prove that $(U_1^{(r)})^n = U_n^{(r)}$, $n = 1, 2, \dots$;

(C) Let there be r sequences $(H_{n,j}^{(r)})_{n=0}^{\infty}$ satisfying the common recurrence in (2), but the arbitrary initial values. Form the $r \times r$ matrix

$$H_n^{(r)} = ((H_{n+r-i,j}^{(r)})).$$

Prove that

$$(U_1^{(r)})^{n-1} H_1^{(r)} = H_n^{(r)}, \quad n = 1, 2, \dots .$$

SOLUTIONS

Woops

H-451 *Proposed by T. V. Padmakumar, Trivandrum, South India (Vol. 29, no. 1, February 1991)*

If p is a prime and x and a are positive integers, show

$$\binom{x + ap}{p} - \binom{x}{p} \equiv a \pmod{p}.$$

Editorial Note: Many readers pointed out that this problem was published in an earlier issue of this Quarterly as B-643. Also, this result readily follows from B-666. In spite of this, we offer one more solution.

Solution by Guo-Gang Gao, University of Montreal, Montreal, Canada

Lemma 1: Let z be a positive integer. If $z + 1 \not\equiv 0 \pmod{p}$, then

$$\binom{z}{p-1} \equiv 0 \pmod{p}.$$

Proof: If $z + 1 \not\equiv 0 \pmod{p}$, then *only* one of $z, z - 1, \dots, z - p + 2$ must be divisible by p, by the pigeonhole principle. Hence, $\binom{z}{p-1}$ always contains a factor of p because p is a prime, and the lemma follows. □

Lemma 2: Let z be a positive integer. Then, for $1 \le k \le p - 1$,

$$\binom{zp - k - 1}{p - k} \equiv 0 \pmod{p}.$$

Proof: Since $zp - k - 1 - (p - k) = (z - 1)p - 1$, $zp - k - 1 \ge (z - 1)$, and $0 < p - k < p$, thus

$$\binom{zp - k - 1}{p - k} = \frac{(zp - k - 1)!}{(zp - k - 1 - p + k)!(p - k)!}$$

always contains a factor of p, i.e., the lemma follows. □

Lemma 3: Let z be a positive integer. Then

$$\binom{zp-1}{p-1} \equiv 1 \pmod{p}.$$

Proof: (a) If $z = 1$, it is trivial; (b) let $z > 1$, then by repetitively applying Lemma 2, and

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

we have

$$\begin{aligned}\binom{zp-1}{p-1} &\equiv \binom{zp-2}{p-1} \pmod{p} + \binom{zp-2}{p-2} \pmod{p}\\ &\equiv \binom{zp-3}{p-2} \pmod{p} + \binom{zp-3}{p-3} \pmod{p}\\ &\vdots\\ &\equiv \binom{zp-p}{0} \pmod{p}\\ &\equiv 1 \pmod{p}.\end{aligned}$$

We now come to the proof of the statement. By repetitively applying

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

we have

$$\begin{aligned}\binom{x+ap}{p} - \binom{x}{p} &= \binom{x+(a-1)p}{p} - \binom{x}{p} + \sum_{i=(a-1)p}^{ap-1}\binom{x+i}{p-1}\\ &\vdots\\ &= \sum_{j=1}^{a}\sum_{i=(j-1)p}^{jp-1}\binom{x+i}{p-1}.\end{aligned}$$

For any fixed j $(1 \le j \le a)$, $x + i$ can be one of p consecutive integers, $x + (j-1)p, \ldots, x + jp - 1$. Of these p consecutive integers, there always exists *only* one $x + i$ such that $x + i + 1 \equiv 0 \pmod{p}$, by the pigeonhole principle. Therefore, by Lemmas 1 and 3, for any fixed j,

$$\sum_{i=(j-1)p}^{jp-1}\binom{x+i}{p-1} \equiv 1 \pmod{p},$$

that is,

$$\binom{x+ap}{p} - \binom{x}{p} \equiv a \pmod{p},$$

completing the proof. □

Also solved by K. Atanassov, P. Bruckman, P. Filipponi, R. Hendel, J. Kostal, Y. H. H. Kwong, B. Prielipp, H.-J. Seiffert, and the proposer.

Divide and Conquer

H-452 *Proposed by Don Redmond, Southern Illinois U., Carbondale, IL (Vol. 29, no. 2, May 1991)*

Let $p_r(m)$ denote the m^{th} r-gonal number $(m/2)\{2 + (r-2)(m-1)\}$. Characterize the values of r and m such that

$$p_r(m) \Big| \sum_{k=1}^{m} p_r(k).$$

Solution by C. Georghiou, University of Patras, Patras, Greece

Let $S_r(m) := \sum_{k=1}^{m} p_r(m)$. Then it is easy to see that

$$S_r(m) = \frac{m(m+1)}{12}[(r-2)(2m+1) - 3(r-4)].$$

Now, since $p_r(1) = 1$ and $S_r(1) = 1$, the given property is trivially true for all r and $m = 1$. So, we are interested in the case $m > 1$ (and, of course, $r > 1$). Then the given property is true only if

$$r = 2 \text{ and } m \equiv 1 \bmod 2 \quad \text{or} \quad r = 3 \text{ and } m \equiv 1 \bmod 3.$$

Indeed, we have

$$S_2(m)/p_2(m) = (m+1)/2 \quad \text{and} \quad S_3(m)/p_3(m) = (m+2)/3.$$

It remains to show that $p_r(m) \nmid S_r(m)$ for $r > 3$ (and $m > 1$). We have

$$S_r(m)/p_r(m) = \frac{(m+1)[(r-2)m - (r-5)]}{3[(r-2)m - (r-4)]}.$$

Since 3 must divide either factor of the numerator, we have the following three possibilities: (i) $m = 3n - 1$; (ii) $m = 3n + 1$; (iii) $r = 3s - 1$ and $m = 3n$.

In Case (i), we get

$$S_r(m)/p_r(m) = n + n/[(3r-6)n - (2r-6)],$$

and since $0 < n/[(3r-6)n - (2r-6)] < 1$ for $n > 0$ and $r > 3$ we conclude that $p_r(3n-1) \nmid S_r(3n-1)$ for any $r > 3$ and any $n > 0$.

In Case (ii), we get

$$S_r(m)/p_r(m) = n + [(2r-3)n + 2]/[(3r-6)n + 2],$$

and it is easy to see that the second term lies (strictly) between 0 and 1 for $r > 3$ and $n > 0$.

Finally, in Case (iii), we get

$$S_r(m)/p_r(m) = n + [3s-2)n - (s-2)]/[(9s-9)n - (3s-5)],$$

and again the second term is positive and less than unity for any $n > 0$ and $s > 1$.

Also solved or partially solved by P. Bruckman, N. Jensen, S. Rabinowitz, and the proposer.

Sum Formulae!

H-453 *Proposed by James E. Desmond, Pensacola Jr. College, Pensacola, FL (Vol. 29, no. 2, May 1991)*

Show that for positive integers m and n,

$$\frac{L_{(2m+1)n}}{L_n} = \sum_{j=1}^{m} (-1)^{(n+1)(m-j)} L_{2nj} + (-1)^{m(n+1)}$$

and

$$\frac{F_{2mn}}{L_n} = \sum_{j=1}^{m} (-1)^{(n+1)(m-j)} F_{n(2j-1)}.$$

Solution by Stanley Rabinowitz, Westford, MA

Lemma:

$$S(n,\ a,\ b,\ r)\ \equiv \sum_{j=1}^{n} r^j F_{aj+b} = \frac{(-1)^a r^{n+2} F_{an+b} - r^{n+1} F_{a(n+1)b} - (-1)^a r^2 F_b + r F_{a+b}}{(-1)^a r^2 - rL_a + 1}.$$

Proof: Let

$$G(x,\ n)\ \equiv \sum_{j=1}^{n} x^j = x\left(\frac{x^n - 1}{x - 1}\right).$$

Now

$$r^j F_{aj+b} = r^j\left(\frac{\alpha^{aj+b} - \beta^{aj+b}}{\sqrt{5}}\right) = \frac{\alpha^b}{\sqrt{5}}(r\alpha^a)^j - \frac{\beta^b}{\sqrt{5}}(r\beta^a)^j.$$

Thus,

$$\begin{aligned}
S(n,\ a,\ b,\ r) &= \frac{\alpha^b}{\sqrt{5}} G(r\alpha^a,\ n) - \frac{\beta^b}{\sqrt{5}} G(r\beta^a,\ n) \\
&= \frac{\alpha^b}{\sqrt{5}}\, r\alpha^a\left(\frac{r^n\alpha^{an} - 1}{r\alpha^a - 1}\right) - \frac{\beta^b}{\sqrt{5}}\, r\beta^a\left(\frac{r^n\beta^{an} - 1}{r\beta^a - 1}\right) \\
&= \frac{r}{\sqrt{5}}\left[\alpha^{a+b}\left(\frac{r^n\alpha^{an} - 1}{r\alpha^a - 1}\right) - \beta^{a+b}\left(\frac{r^n\beta^{an} - 1}{r\beta^a - 1}\right)\right] \\
&= \frac{r}{\sqrt{5}}\left[\frac{\alpha^{a+b}(r\beta^a - 1)(r^n\alpha^{an} - 1) - \beta^{a+b}(r\alpha^a - 1)(r^n\beta^{an} - 1)}{(r\alpha^a - 1)(r\beta^a - 1)}\right] \\
&= \frac{r}{\sqrt{5}}\left[\frac{\begin{gathered} r^{n+1}(\beta^a\alpha^{a(n+1)+b} - \alpha^a\beta^{a(n+1)+b}) - r^n(\alpha^{a(n+1)+b} - \beta^{a(n+1)+b}) \\ - r(\alpha^{a+b}\beta^a - \alpha^a\beta^{a+b}) + \alpha^{a+b} - \beta^{a+b} \end{gathered}}{r^2(\alpha\beta)^a - r(\alpha^a + \beta^a) + 1}\right] \\
&= \frac{r}{\sqrt{5}}\left[\frac{\begin{gathered} r^{n+1}(\alpha\beta)^a(\alpha^{an+b} - \beta^{an+b}) - r^n(\alpha^{a(n+1)+b} - \beta^{a(n+1)+b}) \\ - r(\alpha\beta)^a(\alpha^b - \beta^b) + (\alpha^{a+b} - \beta^{a+b}) \end{gathered}}{(\alpha\beta)^a r^2 - r(\alpha^a + \beta^a) + 1}\right] \\
&= r\left[\frac{r^{n+1}(-1)^a F_{an+b} - r^n F_{a(n+1)+b} - r(-1)^a F_b + F_{a+b}}{(-1)^a r^2 - rL_a + 1}\right] \\
&= \frac{(-1)^a r^{n+2} F_{an+b} - r^{n+1} F_{a(n+1)+b} - (-1)^a r^2 F_b + r F_{a+b}}{(-1)^a r^2 - rL_a + 1}
\end{aligned}$$

which was to be proved.

Using this lemma, we have

$$\begin{aligned}
&\sum_{j=1}^{m} (-1)^{(n+1)(m-j)} F_{n(2j-1)} \\
&= (-1)^{(n+1)m} S(m,\ 2n,\ -n,\ (-1)^{n+1}) \\
&= (-1)^{(n+1)m}\, \frac{(-1)^{(n+1)(m+2)} F_{2mn-n} - (-1)^{(n+1)(m+1)} F_{2n(m+1)-n} - F_{-n} + (-1)^{n+1} F_n}{2 - (-1)^{n+1} L_{2n}} \\
&= \frac{F_{n(2m-1)} + (-1)^n F_{n(2m+1)}}{2 + (-1)^n L_{2n}}
\end{aligned}$$

where we have used the fact that $F_{-n} = (-1)^{n+1} F_n$.

Thus, it remains to prove that our answer,

$$(1) \qquad \sum_{j=1}^{m} (-1)^{(n+1)(m-j)} F_{n(2j-1)} = \frac{F_{n(2m-1)} + (-1)^n F_{n(2m+1)}}{2 + (-1)^n L_{2n}}$$

is equivalent to the proposer's answer of F_{2mn}/L_n. Cross multiplying, we see that this would be equivalent to showing that

$$(2)\qquad F_{n(2m-1)}L_n + (-1)^n F_{n(2m+1)} = 2F_{2mn} + (-1)^n F_{2mn}L_{2n}.$$

Applying the well-known identity,

$$F_x L_y = F_{x+y} + (-1)^y F_{x-y}$$

to equation (2), we find that all the terms drop out; hence, equation (2) is true. Thus, our answer (1) is equivalent to the proposer's answer.

In the same manner, we can prove a similar lemma for the Lucas numbers:

$$\begin{aligned}
T(n, a, b, r) &\equiv \sum_{j=1}^{n} r^j L_{aj+b} \\
&= \alpha^b G(r\alpha^a, n) + \beta^b G(r\beta^a, n) \\
&= \alpha^b r\alpha^a\left(\frac{r^n\alpha^{an} - 1}{r\alpha^a - 1}\right) + \beta^b r\beta^a\left(\frac{r^n\beta^{an} - 1}{r\beta^a - 1}\right) \\
&= r\left[\alpha^{a+b}\left(\frac{r^n\alpha^{an} - 1}{r\alpha^a - 1}\right) + \beta^{a+b}\left(\frac{r^n\beta^{an} - 1}{r\beta^a - 1}\right)\right] \\
&= r\left[\frac{\alpha^{a+b}(r\beta^a - 1)(r^n\alpha^{an} - 1) + \beta^{a+b}(r\alpha^a - 1)(r^n\beta^{an} - 1)}{(r\alpha^a - 1)(r\beta^a - 1)}\right] \\
&= r\left[\frac{\begin{array}{c} r^{n+1}(\beta^a\alpha^{a(n+1)+b} + \alpha^a\beta^{a(n+1)+b}) - r^n(\alpha^{a(n+1)+b} + \beta^{a(n+1)+b}) \\ - r(\alpha^{a+b}\beta^a + \alpha^a\beta^{a+b}) + (\alpha^{a+b} + \beta^{a+b}) \end{array}}{r^2(\alpha\beta)^a - r(\alpha^a + \beta^a) + 1}\right] \\
&= r\left[\frac{\begin{array}{c} r^{n+1}(\alpha\beta)^a(\alpha^{an+b} + \beta^{an+b}) - r^n(\alpha^{a(n+1)+b} + \beta^{a(n+1)+b}) \\ - r(\alpha\beta)^a(\alpha^b + \beta^b) + (\alpha^{a+b} + \beta^{a+b}) \end{array}}{(\alpha\beta)^a r^2 - r(\alpha^a + \beta^a) + 1}\right] \\
&= \frac{(-1)^a r^{n+2}L_{an+b} - r^{n+1}L_{a(n+1)+b} - (-1)^a r^2 L_b + rL_{a+b}}{(-1)^a r^2 - rL_a + 1}.
\end{aligned}$$

Using this result, we have

$$\begin{aligned}
&\sum_{j=1}^{m} (-1)^{(n+1)(m-j)} L_{2nj} \\
&= (-1)^{(n+1)m} T(m, 2n, 0, (-1)^{n+1}) \\
&= (-1)^{(n+1)m}\left[\frac{(-1)^{(n+1)(m+2)}L_{2mn} - (-1)^{(n+1)(m+1)}L_{2n(m+1)} - L_0 + (-1)^{n+1}L_{2n}}{2 - (-1)^{n+1}L_{2n}}\right] \\
&= \frac{L_{2mn} + (-1)^n L_{2n(m+1)} - 2(-1)^{(n+1)m} + (-1)^{(n+1)(m+1)}L_{2n}}{2 + (-1)^n L_{2n}}.
\end{aligned}$$

To show that our answer is equivalent to the proposer's, we must show that

$$\frac{L_{(2m+1)n}}{L_n} - (-1)^{m(n+1)} = \frac{L_{2mn} + (-1)^n L_{2n(m+1)} - 2(-1)^{(n+1)m} + (-1)^{(n+1)(m+1)}L_{2n}}{2 + (-1)^n L_{2n}}$$

or, equivalently,

$$2L_{n(2m+1)} - 2(-1)^{m(n+1)}L_n + (-1)^n L_{2n}L_{n(2m+1)} - (-1)^{m(n+1)+n}L_nL_{2n}$$
$$= L_nL_{2mn} + (-1)^n L_nL_{2n(m+1)} - 2(-1)^{m(n+1)}L_n + (-1)^{(n+1)(m+1)} L_nL_{2n}.$$

Again, this falls out by applying the well-known identity,

$$L_xL_y = L_{x+y} + (-1)^y L_{x-y}.$$

Also solved by P. Bruckman, N. Jensen, B. Prielipp, H.-J. Seiffert, and the proposer.

Editorial Note: Several readers have pointed out that H-462 was published earlier as H-449.

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