ON SEQUENCES HAVING SAME MINIMAL ELEMENTS IN THE LEMOINE-KATAI ALGORITHM

Jukka Pihko
University of Helsinki, Hallituskatu 15, SF-00100 Helsinki, Finland
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1. Introduction

Let $1 = a_1 < a_2 < \cdots$ be an infinite strictly increasing sequence of positive integers. Let n be a positive integer. We write

(1.1) $\quad n = a_{(1)} + a_{(2)} + \cdots + a_{(s)},$

where $a_{(1)}$ is the greatest element of the sequence $\le n$, $a_{(2)}$ is the greatest element $\le n - a_{(1)}$, and, generally, $a_{(i)}$ is the greatest element $\le n - a_{(1)} - a_{(2)} - \cdots - a_{(i-1)}$. This algorithm for additive representation of positive integers was introduced in 1969 by Kátai ([2], [3], [4]). Lemoine had earlier considered the special cases $a_i = i^k$, $k \ge 2$ ([5], [6]), and $a_i = i(i+1)/2$ ([7]). (See [10] for further information and note also [1].) The above algorithm is, in turn, a special case of a more general algorithm introduced by Nathanson ([9]) in 1975.

The following basic definitions and results are taken from [8] and [10]. We denote here the set of positive integers by $\mathbf{N}$.

Let $1 = a_1 < a_2 < \cdots$ be an infinite strictly increasing sequence of positive integers with the first element equal to 1. We call it an *A-sequence* and denote by A the sequence itself or sometimes the set consisting of the elements of the sequence. We denote the number s of terms in (1.1) by $h(n)$. If the set $\{n \in \mathbf{N} \mid h(n) = m\}$ is nonempty for some $m \in \mathbf{N}$, we say that *y_m exists* and define y_m to be the *smallest* element of this set. If y_m exists for every $m \in \mathbf{N}$, we say that the *Y-sequence exists* and we denote the sequence $1 = y_1 < y_2 < \cdots$ by Y. The elements y_m are also called *minimal elements*.

Theorem 1.1 (Lord): Let y_k be given ($k \in \mathbf{N}$). Then y_{k+1} exists if and only if there exists a number $n \in \mathbf{N}$ such that

$$a_{n+1} - a_n - 1 \ge y_k.$$

Furthermore, if y_{k+1} exists, then $y_{k+1} = y_k + a_m$, where m is the smallest number in the set

$$\{n \in \mathbf{N} \mid a_{n+1} - a_n - 1 \ge y_k\}.$$

Proof: [8], [10, p. 9]. □

It follows that the Y-sequence exists if and only if the set

$$\{a_{n+1} - a_n \mid n \in \mathbf{N}\}$$

is not bounded.

For technical reasons, we sometimes wish to start the A-sequences and Y-sequences with an element $a_0 = 0$ or $y_0 = 0$, respectively. The following result is from [10, p. 14].

Theorem 1.2: Suppose that $B: 0 = b_0 < 1 = b_1 < b_2 < \cdots$ is an infinite sequence of nonnegative integers. Then B is the Y-sequence for some A-sequence if and only if it satisfies the following conditions:

(a) For every $n \in \mathbf{N}$, either

(1) $b_{n+1} - b_n = b_n - b_{n-1}$, or
(2) $b_{n+1} \geq 2b_n + 1$.

(b) The condition (2) in (a) holds for infinitely many $n \in \mathbf{N}$.

In section 2 of this paper we determine, given a sequence B satisfying the conditions (a) and (b) above, *all* A-sequences A such that $Y = B$ (Theorem 2.1). In section 3 we establish *how many* such A-sequences there are (Theorem 3.5). Fibonacci numbers make their appearance there (after Definition 3.1). For other connections of Fibonacci numbers with the Lemoine-Kátai algorithm we refer to [11] and especially to [12], which also provides part of the motivation for this paper.

2. Determination of All A-Sequences Having a Given Y-Sequence

Theorem 2.1: Let the sequence $B: 0 = b_0 < 1 = b_1 < b_2 < \cdots$ satisfy the conditions (a) and (b) of Theorem 1.2. For the A-sequence $A: 1 = a_1 < a_2 < \cdots$, we have $Y = B$ if and only if the following conditions hold:

(a) $A \cap [b_1, b_2] = \{1, 2, \ldots, b_2 - 1\}$.

(b) Let $n > 1$. If $b_{n+1} - b_n = b_n - b_{n-1}$, then $A \cap [b_n, b_{n+1}] = \emptyset$.

(c) Let $n > 1$. If $b_{n+1} \geq 2b_n + 1$, then $A \cap [b_n, b_{n+1}] = \{a_s, \ldots, a_t\}$, where $a_s < \cdots < a_t$, and

(2.1) $b_n + 1 \leq a_s \leq 2b_n - b_{n-1}$,

(2.2) $a_{i+1} - a_i \leq b_n$, $i = s, \ldots, t - 1$ (if $t > s$),

(2.3) $a_t = b_{n+1} - b_n$.

Proof: The "if" part can be proved in almost exactly the same fashion as the corresponding part in the proof of Theorem 1.2. In fact, we only have to suppress "= 0" on page 16, line 7 in [10]. Notice also that the condition

$$a_s \leq 2b_n - b_{n-1}$$

in (2.1) means that (2.2) holds also for $i = s - 1$. To see this, observe that

(2.4) $a_{s-1} = b_n - b_{n-1}$,

which follows easily using conditions (a), (b), and (c).

To prove the "only if" part we suppose now that $A: 1 = a_1 < a_2 < \cdots$ is an A-sequence such that $Y = B$. We must prove that conditions (a), (b), and (c) hold. Condition (a) is trivial. Let $n > 1$ and suppose that

$$b_{n+1} - b_n = b_n - b_{n-1}.$$

From our definitions, it follows easily that

(2.5) $A \cap B = \{1\}$.

Suppose that condition (b) is not true. Then, using (2.5) and $B = Y$, we would get

$$\{y_n + 1, y_n + 2, \ldots, y_n + (y_n - y_{n-1})\} \cap A$$
$$= \{b_n + 1, \ldots, b_{n+1}\} \cap A \neq \emptyset,$$

and so, by [10, Th. 1.13, p. 13],

$$b_{n+1} \geq 2b_n + 1,$$

a contradiction.

Suppose now that $n > 1$ and $b_{n+1} \geq 2b_n + 1$. Suppose further that (a) holds and that (b) and (c) hold for all $n' \in \mathbf{N}$, $1 < n' < n$ if $n > 2$. We prove that (c) holds for n. Since $b_n + 1 \leq b_{n+1} - b_n < b_{n+1}$ and since, by Theorem 1.1, $y_{n+1} - y_n = b_{n+1} - b_n \in A$, we see that

$$A \cap [b_n, b_{n+1}] = \{a_s, \ldots, a_t\}$$

with $a_s < \cdots < a_t$ and $b_{n+1} - b_n = a_h$ for some h, $s \leq h \leq t$. We must prove that $h = t$. By Theorem 1.1 and the definition of h, we get

$$a_{h+1} - a_h - 1 \geq b_n.$$

If $h < t$, then we would get

$$a_{h+1} - a_h - 1 \leq b_{n+1} - (b_{n+1} - b_n) - 1 = b_n - 1 < b_n,$$

a contradiction. It follows that (2.3) holds.

If we had $a_{i+1} - a_i > b_n$ for some i, $s - 1 \leq i \leq t - 1$, then we would have $a_{i+1} - a_i - 1 \geq b_n$ and so, by Theorem 1.1,

$$b_{n+1} \leq b_n + a_i < b_n + a_t = b_{n+1},$$

a contradiction. This proves (2.2). Finally, (2.1) follows from (2.5) and the case $i = s - 1$ above, noticing that using our induction hypothesis we get (2.4) as before. Theorem 2.1 is now proved. □

3. The Number of A-Sequences Having a Given Y-Sequence

Suppose that $B: 0 = b_0 < 1 = b_1 < b_2 < \cdots$ satisfies conditions (a) and (b) of Theorem 1.2. Let $n > 1$ and suppose that $b_{n+1} \geq 2b_n + 1$. Let $I(n)$ be the number of different sequences $a_s < \cdots < a_t$ satisfying conditions (2.1), (2.2), and (2.3). We are going to evaluate $I(n)$. For that, we need the following

Definition 3.1: Let $j \in \mathbf{N}$. Let $u_i^{(j)}$, $i = 1, 2, \ldots$, be such that

$$u_i^{(j)} = \begin{cases} 2^{i-1} & \text{for } i = 1, 2, \ldots, j, \\ u_{i-1}^{(j)} + \cdots + u_{i-j}^{(j)} & \text{for } i > j. \end{cases}$$

In particular, we have $u_i^{(1)} = 1$, $i = 1, 2, \ldots$, and $u_i^{(2)} = F_{i+1}$, $i = 1, 2, \ldots$ (where F_{i+1} denotes the Fibonacci number).

Lemma 3.2: Let $a, b \in \mathbf{Z}$, $a < b$, $j \in \mathbf{N}$. The number of all possible sets $\{c_1, \ldots, c_k\}$ (k is *not* fixed), where

$$a = c_1 < c_2 < \cdots < c_k = b, \; c_i \in \mathbf{Z}, \; i = 1, \ldots, k,$$

and

$$c_{i+1} - c_i \leq j, \; i = 1, \ldots, k - 1,$$

is $u_{b-a}^{(j)}$.

Proof: If $b - a \leq j$, then any subset of the set $\{a + 1, \ldots, b - 1\}$, arranged as a sequence $c_2 < \cdots < c_{k-1}$, gives rise to a permissible sequence

$$a = c_1 < c_2 < \cdots < c_k = b.$$

There are $b - a - 1$ members in the set $\{a + 1, \ldots, b - 1\}$.

If $b - a > j$, then c_2 must be one of the numbers $a + 1, a + 2, \ldots, a + j$, and we use induction. □

Theorem 3.3: Let $n > 1$ and $b_{n+1} \geq 2b_n + 1$.

(a) $I(n) = 2^{b_{n+1} - 2b_n - 1}$, if $2b_n - b_{n-1} \geq b_{n+1} - b_n$.

(b) $I(n) = \sum_{i=g}^{h} u_i^{(b_n)}$, if $2b_n - b_{n-1} < b_{n+1} - b_n$, where

$$g = b_{n+1} - 3b_n + b_{n-1} \quad \text{and} \quad h = b_{n+1} - 2b_n - 1.$$

(c) In case (b), if $(b_{n+1} - b_n) - (b_n + 1) \le b_n$, then

$$I(n) = 2^{b_{n+1} - 2b_n - 1} - 2^{b_{n+1} - 3b_n + b_{n-1} - 1}.$$

Proof: These results follow easily from Theorem 2.1, the definition of $I(n)$, and the use of Lemma 3.2. □

Corollary 3.4: Let $n > 1$ and $b_{n+1} \ge 2b_n + 1$. We have $I(n) = 1$ if and only if

(a) $b_{n+1} = 2b_n + 1$, or

(b) $b_{n+1} = 2b_n + 2$ and $b_n = b_{n-1} + 1$.

Proof: The "if" part is clear. To prove the "only if" part, we suppose that neither (a) nor (b) holds. Then we must have $b_{n+1} \ge 2b_n + 2$.

(1) If $b_{n+1} = 2b_n + 2$, we must have $b_n - b_{n-1} \ge 2$. It follows that

$$2b_n - b_{n-1} \ge b_n + 2 = b_{n+1} - b_n.$$

According to Theorem 3.3, we have

$$I(n) = 2^{b_{n+1} - 2b_n - 1} = 2^{2-1} = 2.$$

(2) Let $b_{n+1} \ge 2b_n + 3$. If $2b_n - b_{n-1} \ge b_{n+1} - b_n$, then, according to Theorem 3.3, we have

$$I(n) = 2^{b_{n+1} - 2b_n - 1} \ge 2^{3-1} = 4.$$

On the other hand, if $2b_n - b_{n-1} < b_{n+1} - b_n$, then, again by Theorem 3.3,

$$I(n) \ge u_h^{(b_n)} = u_{b_{n+1} - 2b_n - 1}^{(b_n)} \ge u_{3-1}^{(b_n)} = u_2^{(b_n)} > 1.$$

In the last inequality, we use the fact that $b_n > 1$, which follows from $n > 1$, and the proof is complete. □

Theorem 3.5: Let $B: 0 = b_0 < 1 = b_1 < b_1 < \cdots$ be an infinite sequence of non-negative integers satisfying the conditions (a) and (b) of Theorem 1.2. Let $I(B)$ denote the number of different A-sequences for which $Y = B$. Then $I(B)$ is finite if and only if there exists $n_0 \in \mathbb{N}$ such that $b_{n+1} \le 2b_n + 1$ for all $n \ge n_0$. In that case

$$I(B) = \prod_{\substack{1 \le n \le n_0 \\ b_{n+1} \ge 2b_n + 1}} I(n) \quad [\text{we define } I(1) = 1]. \tag{3.1}$$

Proof: From Theorem 2.1 it is clear that $I(B)$ is finite if and only if for some point on we always have $I(n) = 1$ for n satisfying $b_{n+1} \ge 2b_n + 1$. From Corollary 3.4 we know exactly when $I(n) = 1$. It remains to observe that condition (b) of Corollary 3.4 can hold for at most one n. □

Examples 3.6:

(a) ([10, p. 16], [12, p. 296]) Let B be defined by $b_0 = 0$, $b_{n+1} = 2b_n + 1$, $n = 0, 1, \ldots$. Then $b_n = 2^n - 1$ for every $n \in \mathbb{N}$ and by (3.1) we get $I(B) = 1$. The only A-sequence A satisfying $Y = B$ is given by $a_n = 2^{n-1}$, $n = 1, 2, \ldots$.

(b) Let us modify the example given above by taking B: 0, 1, 3, 10, 17, 24, 31, 63, 127, ..., $2^n - 1$, ... Using (3.1) and Theorem 3.3 [we can use (b) or (c)], we get $I(B) = I(2) = 6$. The six A-sequences for which $Y = B$ are given by

1, 2, 4, 5, 6, 7, 32, 64, ..., 2^n, ...,
1, 2, 4, 6, 7, 32, 64, ..., 2^n, ...,
1, 2, 4, 5, 7, 32, 64, ..., 2^n, ...,
1, 2, 4, 7, 32, 64, ..., 2^n, ...,
1, 2, 5, 6, 7, 32, 64, ..., 2^n, ...,
1, 2, 5, 7, 32, 64, ..., 2^n, ...

(c) We modify the examples given above and take $B: 0, 1, 3, 17, 31, 63, 127, \ldots$. We again obtain $I(B) = I(2)$. This time we have to use part (b) of Theorem 3.3 to calculate $I(2)$. The result is

$$I(B) = I(2) = u_9^{(3)} + u_{10}^{(3)} = 149 + 274 = 423.$$

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References

1. A. S. Fraenkel. "Systems of Numeration." *Amer. Math. Monthly* *92* (1985):105-114.
2. I. Kátai. "Some Algorithms for the Representation of Natural Numbers." *Acta Sci. Math.* (Szeged) *30* (1969):99-105.
3. I. Kátai. "On an Algorithm for Additive Representation of Integers by Prime Numbers." *Ann. Univ. Sci. Budapest, Eötvös Sect. Math.* *12* (1969):23-27.
4. I. Kátai. "On Additive Representation of Integers." *Ann. Univ. Sci. Budapest, Eötvös Sect. Math.* *13* (1970):77-81.
5. E. Lemoine. "Décomposition d'un nombre entier N en ses puissances n$^{\text{ièmes}}$ maxima." *C. R. Paris* *XCV* (1882):719-22.
6. E. Lemonie. "Sur la décomposition d'un nombre en ses carrés maxima." *Assoc. Franc. Tunis* *25* (1896):73-77.
7. E. Lemoine. "Note sur deus nouvèles décompositions des nombres entiers." *Assoc. Franc. Paris* *29* (1900):72-74.
8. G. Lord. "Minimal Elements in an Integer Representing Algorithm." *Amer. Math. Monthly* *83* (1976):193-95.
9. M. B. Nathanson. "An Algorithm for Partitions." *Proc. Amer. Math. Soc.* *52* (1975):121-24.
10. J. Pihko. "An Algorithm for Additive Representation of Positive Integers." *Ann. Acad. Sci. Fenn. Ser. A. I Math. Dissertationes No.* *46* (1983):1-54.
11. J. Pihko. "On Fibonacci and Lucas Representations and a Theorem of Lekkerkerker." *Fibonacci Quarterly* *26.3* (1988):256-61.
12. J. Pihko. "Fibonacci Numbers and an Algorithm of Lemoine and Kátai." In *Applications of Fibonacci Numbers*. Ed. G. E. Bergum et al. Kluwer: Academic Publishers, 1990, pp. 287-97.

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