FORMAL POWER SERIES FOR BINOMIAL SUMS OF SEQUENCES OF NUMBERS

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1. INTRODUCTION

Let $\{A_n\}_{n=0}^{\infty}$ be a given sequence of numbers and let

$$S(n)=\sum_{k=0}^{n}\binom{n}{k}A_k,\quad n=0,1,\ldots$$

Let $A(x)$ and $S(x)$ denote the formal power series determined by the sequences $\{A_n\}_{n=0}^{\infty}$ and $\{S(n)\}_{n=0}^{\infty}$, that is,

$$A(x)=\sum_{n=0}^{\infty}A_nx^n,\quad S(x)=\sum_{n=0}^{\infty}S(n)x^n.$$

Recently, H. W. Gould [2] pointed out that

$$S(x)=\frac{1}{1-x}A\left(\frac{x}{1-x}\right). \tag{1}$$

In this paper we shall give a straightforward generalization of (1) and an application and a modification of the generalization.

2. A GENERALIZATION

Let s, t be given complex numbers and let $\{A_n\}_{n=0}^{\infty}$ be a given sequence of numbers. Denote

$$S(n)=\sum_{k=0}^{n}\binom{n}{k}t^{n-k}s^kA_k,\quad n=0,1,\ldots$$

Theorem 1: We have

$$S(x)=\frac{1}{1-tx}A\left(\frac{sx}{1-tx}\right).$$

Proof: The proof is similar to that of (1) given in [2]. In fact,

$$\begin{aligned}S(x)&=\sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}\binom{n}{k}t^{n-k}s^kA_k=\sum_{k=0}^{\infty}A_ks^kx^k\sum_{n=k}^{\infty}\binom{n}{k}t^{n-k}x^{n-k}\\&=\sum_{k=0}^{\infty}A_ks^kx^k(1-tx)^{-k-1}=(1-tx)^{-1}A\left(\frac{sx}{1-tx}\right).\end{aligned}$$

This completes the proof.

3. AN APPLICATION

Let F_n, $n = 0, 1, \ldots$, be the Fibonacci numbers, and take $F_{-n} = (-1)^{n-1} F_n$. Let p, q be fixed nonzero integers such that $p \neq q$, and let r be a fixed integer. L. Carlitz [1, Theorem 4] proved that

$$\lambda^n F_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk+r} \quad (n = 0, 1, \ldots)$$

if and only if

$$\lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}.$$

We shall apply Theorem 1 to give a proof for this result. This result is given in Theorem 2 and in a slightly different form.

Lemma 1: We have

$$\sum_{n=0}^{\infty} F_{pn+r} x^n = \frac{F_r + (-1)^r F_{p-r} x}{1 - L_p x + (-1)^p x^2},$$

where L_n is the n^{th} Lucas number and $L_{-n} = (-1)^n L_n$ for $n \geq 0$.

Lemma 2: We have

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k F_{qk+r} = \frac{F_r + ((-1)^r F_{q-r} s - F_q t)x}{1 - (2t + L_q s)x + (t^2 + L_q ts + (-1)^q s^2)x^2}.$$

Lemma 1 is the same as formula (6) of [3]. Lemma 2 follows from Theorem 1 and Lemma 1.

Theorem 2: We have

$$\sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k F_{qk+r} = F_{pn+r} \quad (n = 0, 1, \ldots) \tag{2}$$

if and only if

$$s = F_p / F_q, \quad t = (-1)^p F_{q-p} / F_q. \tag{3}$$

Proof: By Lemmata 1 and 2, (2) holds if and only if,

$$(-1)^r F_{q-r} s - F_q t = (-1)^r F_{p-r}, \tag{4}$$

$$2t + L_q s = L_p, \tag{5}$$

$$t^2 + L_q ts + (-1)^q s^2 = (-1)^p. \tag{6}$$

Solving (4) and (5) gives (3). It can be verified that (6) holds for those values of s and t. This completes the proof.

4. A MODIFICATION

An interesting problem is to find a sequence $\{T(n)\}_{n=0}^{\infty}$ such that

$$(7) \qquad T(x) = A\left(\frac{sx}{1-tx}\right).$$

The solution is simple. It is given in Theorem 3. Applications of (7) are given in Theorem 4 and Theorem 5.

Theorem 3: Let $T(0) = A_0$ and

$$(8) \qquad T(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} t^{n-k} s^k A_k, \ n = 1, 2, \dots$$

Then (7) holds.

Proof: We have

$$T(x) = (1-tx)S(x).$$

Thus, $T(0) = S(0) = A_0$ and for $n \ge 1$,

$$T(n) = S(n) - tS(n-1) = s^n A_n + \sum_{k=0}^{n-1}\left[\binom{n}{k} - \binom{n-1}{k}\right] t^{n-k} s^k A_k$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1} t^{n-k} s^k A_k.$$

This completes the proof.

Remark: Theorem 3 could also be proved in a similar way to Theorem 1.

Theorem 4: If $s \ne 0$ and $T(n)$, $n = 0, 1, \dots$, is given by (8), then $A_0 = T(0)$ and

$$A_n = s^{-n} \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} t^{n-k} T(k), \ n = 1, 2, \dots .$$

Proof: By (7),

$$A(x) = T\left(\frac{x}{s+tx}\right) = T(0) + \sum_{n=0}^{\infty} x^n s^{-n} \sum_{k=1}^{n} \binom{n-1}{k-1} (-t)^{n-k} T(k).$$

This proves Theorem 4.

Let m be a nonnegative integer. Then we define $T_m(n)$, $n = 0, 1, \dots$, inductively by

$$T_0(n) = A_n, \ n = 0, 1, \dots,$$

$$T_{m+1}(0) = A_0, \ T_{m+1}(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} t^{n-k} s^k T_m(n), \ n = 1, 2, \dots$$

when $m \ge 0$.

Theorem 5: If $s \neq 1$, then

$$T_m(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} t^{n-k} \left(\frac{s^m - 1}{s-1} \right)^{n-k} s^{mk} A_k, \; n = 1, 2, \ldots .$$

Proof: Theorem 5 can be proved by applying the formula

$$T_{m+1}(x) = T_m\left(\frac{sx}{1-tx} \right).$$

Remark: The transformations T and T_m have their analogues in the theory of arithmetic functions (see [4]).

REFERENCES

1. L. Carlitz. "Some Classes of Fibonacci Sums." *Fibonacci Quarterly* **16.5** (1978):411-26.
2. H. W. Gould. "Series Transformations for Finding Recurrences for Sequences." *Fibonacci Quarterly* **28.2** (1990):166-71.
3. R. T. Hansen. "General Identities for Linear Fibonacci and Lucas Summations." *Fibonacci Quarterly* **16.2** (1978):121-28.
4. P. Kesava Menon. "Transformations of Arithmetic Functions." *J. Indian Math. Soc.* **6** (1942):143-52.

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