THREE-SQUARE THEOREM AS AN APPLICATION OF ANDREWS' IDENTITY

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(Submitted June 1991)

1. INTRODUCTION

The representation of an integer n as a sum of k squares is one of the most beautiful problems in the theory of numbers. Such representations are useful in lattice point problems, crystallography, and certain problems in mechanics [6, pp. 1-4]. If $r_k(n)$ denotes the number of representations of an integer n as a sum of k squares, Jacobi's two- and four-square theorems [9] are:

(1)
$$r_2(n) = 4[d_1(n) - d_3(n)]$$

and

(2)
$$r_4(n) = 8 \sum_{\substack{d|n \\ d \not\equiv 0 \pmod 4}} d$$

where $d_i(n)$ denotes the number of divisors of n, $d \equiv i \pmod 4$. In literature there are several proofs of (1) and (2). For instance, M. D. Hirschhorn [7; 8] proved (1) and (2) using Jacobi's triple product identity. S. Bhargava & Chandrashekar Adiga [4] have proved (1) and (2) as a consequence of Ramanujan's ${}_1\Psi_1$ summation formula [10]. Recently R. Askey [2] has proved (1) and also derived a formula for the representation of an integer as a sum of a square and twice a square. The authors [5] have derived a formula for the representation of an integer as a sum of a square and thrice a square. These works of Askey [2] and the authors [5] also rely on Ramanujan's ${}_1\Psi_1$ summation [10].

In 1951 P. T. Bateman [3] obtained the following formula for $r_3(n)$:

(3)
$$r_3(n) = \frac{16}{\pi}\sqrt{n}L(1,\chi)q(n)P(n),$$

where
$$n = 4^a n_1, \quad 4 \nmid n_1,$$

$$q(n) = \begin{cases} 0 & \text{if } n_1 \equiv 7 \pmod 8, \\ 2^{-a} & \text{if } n_1 \equiv 3 \pmod 8, \\ 3\cdot 2^{-a-1} & \text{if } n_1 \equiv 1,2,5, \text{ or } 6 \pmod 8, \end{cases}$$

$$P(n) = \prod_{\substack{p^{2b}|n \\ p \text{ odd}}} \left[1 + \sum_{j=1}^{b-1} p^{-j} + p^{-b}\left(1 - \left[\frac{(-n/p^{2b})}{p}\right]\frac{1}{p}\right)^{-1}\right],$$

$$\left(P(n)=1 \text{ for square-free } n\right), \text{ and}$$

$$L(S,\chi)=\sum_{m=1}^{\infty}\chi(m)m^{-S} \text{ with } \chi(m), \text{ the Legendre-Jacobi-Kronecker symbol:}$$

$$\chi(m)=\left(\frac{-4}{m}\right)=\begin{cases}1 & \text{if } m\equiv 1 \pmod 4,\\ 0 & \text{if } m\equiv 0 \pmod 2,\\ -1 & \text{if } m\equiv 3 \pmod 4.\end{cases}$$

In this note we obtain an alternate formula (13) for $r_3(n)$ which involves only partition functions unlike Bateman's formula (3) which is expressed in terms of Dirichlet's series {6, pp. 54, 55]. To derive our formula (13) for $r_3(n)$, we employ G. E. Andrews' [1] generalization of Ramanujan's ${}_1\Psi_1$ summation:

$$\frac{(a^{-1}-b^{-1})(A)_\infty(B)_\infty(bq/a)_\infty(aq/b)_\infty(q)_\infty(AB/ab)_\infty}{(-b)_\infty(-a)_\infty(-A/b)_\infty(-A/a)_\infty(-B/b)_\infty(-B/a)_\infty} \tag{4}$$

$$=a^{-1}\sum_{m=0}^{\infty}\frac{(-q/a)_m(AB/ab)_m(-b)^m}{(-B/a)_{m+1}(-A/a)_{m+1}}-b^{-1}\sum_{m=0}^{\infty}\frac{(A)_m(-aq/B)_m(-B/b)^m}{(-a)_{m+1}(-A/b)_{m+1}},$$

where

$$(a)_\infty=(a;q)_\infty=\prod_{m=0}^{\infty}(1-aq^m)$$

and

$$(a)_m=(a;q)_m=\frac{(a;q)_\infty}{(aq^m;q)_\infty},\quad |q|<1.$$

2. THREE-SQUARE THEOREM

In this section we derive a formula for $r_3(n)$. the number of representations of an integer n as a sum of three squares. For convenience, we first transform Andrews' formula (4).

Lemma 2.1 (G. E. Andrews' [1]):

$$\frac{(A;q^2)_\infty(-A\beta/\alpha qz;q^2)_\infty(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty}{(-A/\alpha qz;q^2)_\infty(A/\alpha q^2;q^2)_\infty(-\alpha qz;q^2)_\infty(-\beta q/z;q^2)_\infty(\alpha q^2;q^2)_\infty(\beta q^2;q^2)_\infty} \tag{5}$$

$$=\frac{1}{[1-(A/\alpha q^2)]}+\sum_{m=1}^{\infty}\frac{(1/\alpha;q^2)_m(-A\beta/\alpha qz;q^2)_m(-\alpha q)^m)}{(\beta q^2;q^2)_m(A/\alpha q^2;q^2)_{m+1}}z^m$$

$$+\sum_{m=1}^{\infty}\frac{(1/\beta;q^2)_m(A;q^2)_{m-1}(-\beta q)^m}{(\alpha q^2;q^2)_m(-A/\alpha qz;q^2)_m}z^{-m}$$

if $|\beta q|<|z|<1/|\alpha q|$ and $|q|<1$ with none of the factors in the denominators of (5) being 0.

Proof: Equation (4) is equivalent to

$$\frac{a^{-1}[1-(a/b)](A)_\infty(B)_\infty(bq/a)_\infty(aq/b)_\infty(q)_\infty(AB/ab)_\infty}{[1+(B/a)](-b)_\infty(-a)_\infty(-A/b)_\infty(-A/a)_\infty(-B/b)_\infty(Bq/a)_\infty}$$

$$=a^{-1}\frac{1}{[1+(B/a)]}\sum_{m=0}^{\infty}\frac{(-q/a)_m(AB/ab)_m(-b)^m}{(-Bq/a)_m(-A/a)_{m+1}}$$

$$-b^{-1}\frac{(-b/B)}{[1+(a/B)]}\sum_{m=0}^{\infty}\frac{(A)_m(-a/B)_{m+1}(-b/B)^{-m-1}}{(-a)_{m+1}(-A/b)_{m+1}}$$

which, in turn is equivalent to

$$\frac{(A)_\infty(B)_\infty(bq/a)_\infty(a/b)_\infty(q)_\infty(AB/ab)_\infty}{(-b)_\infty(-a)_\infty(-A/b)_\infty(-A/a)_\infty(-B/b)_\infty(-Bq/a)_\infty} \tag{6}$$

$$=\sum_{m=0}^{\infty}\frac{(-q/a)_m(AB/ab)_m(-b)^m}{(-Bq/a)_m(-A/a)_{m+1}}+\sum_{m=0}^{\infty}\frac{(A)_m(-a/B)_{m+1}(-b/B)^{-m-1}}{(-a)_{m+1}(-A/b)_{m+1}}.$$

Change b to $-z$, a to $-q/a'$, B to b'/a' in (6) to obtain

$$\frac{(A)_\infty(b'/a')_\infty(za')_\infty(q/a'z)_\infty(q)_\infty(Ab'/zq)_\infty}{(z)_\infty(q/a')_\infty(A/z)_\infty(Aa'/q)_\infty(b'/a'z)_\infty(b')_\infty} \tag{7}$$

$$=\sum_{m=0}^{\infty}\frac{(a')_m(Ab'/zq)_m z^m}{(b')_m(Aa'/q)_{m+1}}+\sum_{m=0}^{\infty}\frac{(A)_m(q/b')_{m+1}(a'z/b')^{-(m+1)}}{(q/a')_{m+1}(A/z)_{m+1}}.$$

Change q to q^2, a' to $1/\alpha$, b' to βq^2, and z to $-\alpha qz$ in (7) to obtain (5). Hence, the lemma.

Corollary 2.1:

$$\left(\sum_{n=-\infty}^{\infty}q^{n^2}\right)^3=\frac{(-q;q^2)_\infty^3(q^2;q^2)_\infty^3}{(q;q^2)_\infty^3(-q^2;q^2)_\infty^3} \tag{8}$$

$$=1+2\sum_{m=1}^{\infty}\frac{(-q;q^2)_m q^m}{(1+q^{2m})(-q^2;q^2)_m}+4\sum_{m=1}^{\infty}\frac{(q^2;q^2)_{m-1}q^m}{(1+q^{2m})(q;q^2)_m}.$$

Proof: Putting $\alpha=\beta=-1$, $z=1$, and $A=q^2$ in (5), we have the second of the equations (8), the first being a well-known theta-function identity [10]. In fact, put $z=1$, $A=\alpha=\beta=0$ in (5) and use the easily verified Euler identity

$$(-q;q^2)_\infty=1/(q;q^2)_\infty(-q^2;q^2)_\infty.$$

Before stating the main theorem of this section, we introduce two partition-counting functions $p_m(n)$ and $q_m(n)$.

Definition 2.1: Given a partition π, let $e(\pi)$ denote the number of even parts in π. Define $P_m(n)$ to be the set of partitions of n in which odd parts are distinct and all parts are less than or equal to $2m$, $Q_m(n)$ to be the set of partitions of n in which even parts are distinct and all parts are less than or equal to $2m-1$. We define

(9) $$p_m(n) = \sum_{\pi \in P_m(n)} (-1)^{e(\pi)},$$

(10) $$q_m(n) = \sum_{\pi \in Q_m(n)} (-1)^{e(\pi)},$$

so that

(11) $$\frac{(-q;q^2)_m}{(-q^2;q^2)_m} = \sum_{n=0}^{\infty} p_m(n)q^n,$$

(12) $$\frac{(q^2;q^2)_{m-1}}{(q;q^2)_m} = \sum_{n=0}^{\infty} q_m(n)q^n.$$

Theorem 2.1: If $r_3(n)$ is the number of representations of n as sum of three squares and if $p_m(n)$ and $q_m(n)$ are as defined by (9)-(10), then

(13) $$r_3(n) = \sum_{m=1}^{n} \sum_{0 \le i \le (n-m)/2m} (-1)^i [2p_m(n-2im-m) + 4q_m(n-2im-m)].$$

Proof: Employing (11), (12), and the fact that

$$\frac{q^m}{1+q^{2m}} = \sum_{i=0}^{\infty} (-1)^i q^{2im+m}$$

in (8), we immediately have (13).

ACKNOWLEDGMENT

The authors are thankful to the referee for valuable suggestions which considerably simplified this paper.

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AMS Classification numbers: 11E25, 33D10, 05A17

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NOTICE OF NOVEMBER 1992 VOLUME INDEX CORRECTION

- K. Atanassov's name was inadvertently omitted from the list of authors.
- K. Atanassov's coauthored article "Recurrent Formulas of the Generalized Fibonacci and Tribonacci Sequences" was incorrectly credited to Richard André-Jeannin.