ON THE RECIPROCALS OF THE FIBONACCI NUMBERS

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(Submitted September 1991)

A well-known result concerning partial sums of the reciprocals of the natural numbers $1+1/2+1/3+\cdots+1/n$, is that they never equal an integer (for $n>1$). A similar result concerning partial sums of the Fibonacci numbers, $F_1=1, F_2=1, F_n=F_{n-1}+F_{n-2}$ $(n\geq 3)$, is trivial because

$$3<\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{8}+\frac{1}{13}+\frac{1}{21}+\cdots<4.$$

However, some interesting questions arise if we consider integer multiples of the reciprocals. Specifically, since $F_{m+1}/F_m\geq 1$, the "integer status" of $F_2/F_1+F_3/F_2+\cdots+F_{n+1}/F_n$ is worth investigating $(n\geq 3)$.

Since $(F_n, F_m)=F_{(n,m)}$ [1; Th. VI], the following result tells us that $F_2/F_1+F_3/F_2+\cdots+F_{n+1}/F_n$ is never an integer for $n\geq 3$.

Theorem 1: If $\{c_j\}$ is an arbitrary sequence of integers for which $F_q \nmid c_q$ whenever q is an odd prime, then the sum $c_1/F_1+c_2/F_2+\cdots+c_n/F_n$ can never be an integer for $n\geq 3$.

Proof: If $n\geq 3$, then, by Bertrand's Postulate [2; p. 343], there is at least one odd prime number p in the interval $]n/2, n]$. For $1\leq i\leq n$, let $\widetilde{F}_i=(F_1F_2\cdots F_n)/F_i$. We then have

$$(F_p, \widetilde{F}_i)=\begin{cases} F_p & \text{if } i\neq p \\ 1 & \text{if } i=p \end{cases}$$

because $(F_p, F_j)=F_{(p,j)}=F_1=1$ for $j\neq p$ and $1\leq j\leq n$. Now

$$\frac{c_1}{F_1}+\frac{c_2}{F_2}+\cdots+\frac{c_n}{F_n}=\frac{c_1\widetilde{F}_1+c_2\widetilde{F}_2+\cdots+c_n\widetilde{F}_n}{F_1F_2\cdots F_n}.$$

Since $F_p|F_1F_2\cdots F_n$, $F_p|c_i\widetilde{F}_i$ for $i\neq p$, and $F_p\nmid c_p\widetilde{F}_p$ [by hypothesis and $(F_p, \widetilde{F}_p)=1$], it follows that

$$\frac{c_1\widetilde{F}_1+c_2\widetilde{F}_2+\cdots+c_n\widetilde{F}_n}{F_1F_2\cdots F_n}$$

can never be an integer.

Theorem 1 is a special case of a result that will be stated shortly. Theorem 1 was singled out because it is easily digested and its proof also works in a more general setting.

Let P and Q be relatively prime integers, and let U_n and V_n be the generalized Fibonacci and Lucas sequences, respectively, defined by (see [1] for information on these sequences):

$$U_n=PU_{n-1}-QU_{n-2}, U_0=0, U_1=1 \text{ and } V_n=PV_{n-1}-QV_{n-2}, V_0=2, V_1=P.$$

Since $(U_n, U_m) = U_{(n,m)}$ [1; Th. VI], it seems that we should be able to replace the F's by the U's in Theorem 1 and its proof and have a more general result. This is not the case, however, because $U_m = 0$ and $U_j = \pm 1$ are possibilities for values of m, $j \geq 2$. If we require $P \neq Q$, so that $P = 1 = Q$ and $P = -1 = Q$ are eliminated, then the discussion following Theorem I in [1] tells us that U_n and V_n are nonzero for $n \geq 1$. Thus, we require that $P \neq Q$.

The "revised proof" of Theorem 1 would be invalid if $U_p = \pm 1$. This can happen. In fact, if $P = 2$ and $Q = 3$, then it is easily seen that $U_3 = 1$. Certainly, if $P > 0$ and $Q < 0$, then $U_n > 1$ and $V_n > 1$ for $n > 1$. For other values of P and Q, the situation is not easily resolved; thus, we reflect this in the statement of the general result.

Theorem 2: Let P and Q be chosen so that $|U_q| > 1$ for all odd primes q. If $\{c_j\}$ is an arbitrary sequence of integers for which $U_q \nmid c_q$ whenever q is an odd prime, then the sum $c_1 / U_1 + c_2 / U_2 + \cdots + c_n / U_n$ can never be an integer for $n \geq 3$.

Proof: If we replace F's by U's, $\widetilde{F}$'s by $\widetilde{U}$'s, etc., in the proof of Theorem 1, then we get a proof of the fact that, for $n \geq 3$, $c_1 / U_1 + c_2 / U_2 + \cdots + c_n / U_n$ is never an integer.

The situation is more complicated for the V_i's. For example, if $P = 4$ and $Q = 7$, then $V_1 = 4$, $V_2 = 2$, and $V_3 = -20$, so $1/V_1 + 1/V_2 + (-5)/V_3 = 1$. The following results reveal the source of the complication and a condition that eliminates it.

Recall that $V_n = PV_{n-1} - QV_{n-2}, V_0 = 2, V_1 = P$, and $(P, Q) = 1$.

Lemma 1: If i is a natural number, then $(V_i, P) = P$ when i is odd and $(V_i, P) = (2, P)$ when i is even. Furthermore, if m is odd and j is a natural number that is relatively prime to m, then $(V_m, V_j) = P$ when j is odd, $(V_m, V_j) = (2, P)$ when j is even, and $(P^{-1}V_m, V_j) = 1$ when j is even.

Proof: $(V_i, P) = (PV_{i-1} - QV_{i-2}, P) = (-QV_{i-2}, P) = (V_{i-2}, P)$ [since $(P, Q) = 1$]. This implies that $(V_i, P) = (V_{i-2}, P) = (V_{i-4}, P) = \cdots = (V_1, P) = P$ when i is odd and $(V_i, P) = (V_0, P) = (2, P)$ when i is even.

We now consider natural numbers m and j where m is odd and j is relatively prime to m. Since $(U_{2m}, U_{2j}) = U_{(2m, 2j)} = U_2 = P$ and $U_{2n} = U_n V_n$ for any natural number n, it follows that $P = (U_m V_m, U_j V_j)$. This shows that $(V_m, V_j) | P$. This and the facts that $(V_i, P) = P$ when i is odd and $(V_i, P) = (2, P)$ when i is even imply that $(V_m, V_j) = P$ when j is odd and $(V_m, V_j) = (2, P)$ when j is even. Since $(2, P) = 1$ if P is odd, it follows that $(P^{-1}V_m, V_j) = 1$ when P is odd and j is even. If P is even, then

$$\begin{aligned}(P^{-1}V_m, 2) &= (P^{-1}(PV_{m-1} - QV_{m-2}), 2) = (V_{m-1} - P^{-1}QV_{m-2}, 2)\\ &= (-P^{-1}QV_{m-2}, 2) \;\; [\text{since } (V_{m-1}, 2) = (P, 2) = 2]\\ &= (P^{-1}V_{m-2}, 2) \;\; [(Q, 2) = 1 \text{ since } (Q, P) = 1].\end{aligned}$$

This implies that $(P^{-1}V_m, 2) = (P^{-1}V_1, 2) = 1$. That is, $P^{-1}V_m$ is odd. Thus, $(P^{-1}V_m, V_j) = 1$ also when P is even and j is even.

Remark 1: It is not always true that $(P^{-1}V_m, V_j) = 1$ when j is odd [again, m is an odd natural number and $(m, j) = 1$]. For example, if $P = 6$ and $Q = 1$, then $V_0 = 2, V_1 = 6, V_2 = 34$, $V_3 = 198$, and $(6^{-1}V_3, V_1) = (33,6) = 3$. Actually, one can prove by mathematical induction that there exist integers k_n and r_n such that

$$V_n = \begin{cases} k_n P^3 + nP(-Q)^{(n-1)/2} & \text{if } n \text{ is odd,} \\ r_n P^2 + 2(-Q)^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

This form of V_n shows that and hence $(P^{-1}V_m, V_j) = (m, P)$.

Theorem 3: Let P and Q be chosen so that $|P^{-1}V_q| > 1$ for all odd primes q. If $\{c_j\}$ is an arbitrary sequence of integers for which $P^{-1}V_q \nmid c_q$ whenever q is an odd prime, then the sum $c_1 / V_1 + c_2 / V_2 + \cdots + c_n / V_n$ can never be an integer for $n \geq 3$.

Proof: Let p be an odd prime number in the interval $]n/2, n]$ and let

$$\widetilde{V}_i = \frac{V_1 V_2 \cdots V_n}{V_i} \quad \text{for } 1 \leq i \leq n.$$

Since there are at least $[(n-3)/2]$ odd numbers in the set $\{1, 2, \ldots, i-1, i+1, \ldots, p-1, p+1, \ldots, n\}$ and $(V_k, P) = P$ when k is odd, it follows that

$$V_p P^{[(n-3)/2]} \mid c_i \widetilde{V}_i \quad \text{for } i \neq p.$$

This is not the case for $c_p \widetilde{V}_p$, as we now demonstrate.

$$\begin{aligned} V_p P^{[(n-3)/2]} \mid c_p \widetilde{V}_p &\Leftrightarrow P^{-1} V_p P^{[(n-3)/2]} \mid c_p \frac{V_2 V_3 \cdots V_n}{V_p} \quad (\text{since } V_1 = P) \\ &\Leftrightarrow P^{-1} V_p \mid c_p \frac{P^{-[(n-3)/2]} V_2 V_3 \cdots V_n}{V_p} \end{aligned}$$

Since there are exactly $[(n-3)/2]$ odd numbers in the set $\{2, 3, \ldots, p-1, p+1, \ldots, n\}$,

$$c_p \frac{P^{-[(n-3)/2]} V_2 V_3 \cdots V_n}{V_p} = c_p \left(\prod_{i=1}^{[n/2]} V_{2i} \right) \left(\prod_{\substack{j=1 \\ j \neq (p-1)/2}}^{[n/2]} P^{-1} V_{2j+1} \right).$$

By hypothesis, $P^{-1}V_p \nmid c_p$, and by Lemma 1, $(P^{-1}V_p, V_{2i}) = 1$ and $(P^{-1}V_p, P^{-1}V_{2j+1}) = 1$ (since $2j+1$ is not divisible by p). This implies $V_p P^{[(n-3)/2]} \nmid c_p \widetilde{V}_p$. Thus, as in the proof of Theorem 1, we conclude that $c_1/V_1 + c_2/V_2 + \cdots + c_n/V_n$ can never be an integer for $n \geq 3$.

Corollary 1: If P and Q are chosen so that $|U_q| > 1$ $[|P^{-1}V_q| > 1]$ for all odd primes q, then the sum

$$\frac{U_2}{U_1} + \frac{U_3}{U_2} + \cdots + \frac{U_{n+1}}{U_n} \quad \left[\frac{V_2}{V_1} + \frac{V_3}{V_2} + \cdots + \frac{V_{n+1}}{V_n}\right]$$

can never be an integer for $n \geq 3$.

Proof: If q is an odd prime, then $(U_{q+1}, U_q) = U_1 = 1$ $[(V_{q+1}, P^{-1}V_q) = 1$ by Lemma 1$]$.

Corollary 2: Let k be a fixed positive integer. Let P and Q be chosen so that $|U_{qk}| > |U_k|$ for all odd primes q. If $\{c_j\}$ is an arbitrary sequence of integers for which $U_{qk}U_k^{-1} \nmid c_q$ whenever q is an odd prime, then the sum $c_1/U_k + c_2/U_{2k} + \cdots + c_n/U_{nk}$ can never be an integer for $n \geq 3$.

Remark 2: If α and β are the roots of $x^2 - Px + Q = 0$, then it is well known that

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n.$$

These forms establish the well-known facts that

$$U_{nk} = V_k U_{(n-1)k} - Q^k U_{(n-2)k} \quad \text{and} \quad V_{nk} = V_k V_{(n-1)k} - Q^k V_{(n-2)k}.$$

Furthermore, using $V_{nk} = V_k V_{(n-1)k} - Q^k V_{(n-2)k}$ and mathematical induction, it is easy to see that $V_k | V_{(2i+1)k}$ whenever i is a positive integer. Also, for $k = 2, 3, 4, \ldots$,

$$(V_k, Q) = (PV_{k-1} - QV_{k-2}, Q) = (PV_{k-1}, Q) = (V_{k-1}, Q) = \cdots = (V_1, Q) = 1.$$

Proof of Corollary 2: If $\hat{U}_n = U_{nk}U_k^{-1}$, then $\hat{U}_n = V_k \hat{U}_{n-1} - Q^k \hat{U}_{n-2}$, a generalized Fibonacci sequence, and $|\hat{U}_q| > 1$ for all odd primes q. It then follows from Theorem 2 that, if $\hat{U}_q \nmid c_q$ whenever q is an odd prime, then $c_1/\hat{U}_1 + c_2/\hat{U}_2 + \cdots + c_n/\hat{U}_n$ is never an integer for $n \geq 3$. Thus, if $U_{qk}U_k^{-1} \nmid c_q$ whenever q is an odd prime, then $U_k(c_1/U_k + c_2/U_{2k} + \cdots + c_n/U_{nk})$ is never an integer for $n \geq 3$, and consequently, $c_1/U_k + c_2/U_{2k} + \cdots + c_n/U_{nk}$ is never an integer for $n \geq 3$.

Corollary 3: Let k be a fixed positive integer. Let P and Q be chosen so that $|P^{-1}V_{qk}| > 1$ for all odd primes q. If $\{c_j\}$ is an arbitrary sequence of integers for which $P^{-1}V_{qk} \nmid c_q$ whenever q is an odd prime, then the sum $c_1/V_k + c_2/V_{2k} + \cdots + c_n/V_{nk}$ can never be an integer for $n \geq 3$.

Proof: If $\hat{V}_n = V_{nk}$, then $\hat{V}_n = V_k\hat{V}_{n-1} - Q^k\hat{V}_{n-2}$, a generalized Lucas sequence, and $|P^{-1}\hat{V}_q| > 1$ for all odd primes q. Since $P^{-1}\hat{V}_q \nmid c_q$, the result follows from Theorem 3.

Corollary 4: If $\{c_j\}$ is an arbitrary sequence of integers for which $q \nmid c_q$ whenever q is prime, then the sum $c_1/1 + c_2/2 + \cdots + c_n/n$ can never be an integer for $n \geq 2$.

Proof: If $U_n = n$, then $U_n = 2U_{n-1} - U_{n-2}$. That is, $\{n\}$ is a generalized Fibonacci sequence for which Theorem 2 applies.

Corollary 5: Let P and Q be chosen so that $|U_q| > 1$ $[|P^{-1}V_q| > 1]$ for all odd primes q. If $\{c_j\}$ is an arbitrary sequence of integers for which $U_q \nmid c_q$ $[P^{-1}V_q \nmid c_q]$ whenever q is an odd prime, then the sum $c_1/U_1 + c_2/U_3 + \cdots + c_n/U_{2n-1}$ $[c_1/V_1 + c_2/V_3 + \cdots + c_n/V_{2n-1}]$ can never be an integer for $n \geq 2$.

Proof: Consider the statement of Theorem 2 [Theorem 3] and just take $c_{2j} = U_{2j}$ $[c_{2j} = V_{2j}]$.

Remark 3: Results for U's and V's with even subscripts are special cases of Corollaries 2 and 3.

ACKNOWLEDGMENT

The authors thank the referee for bringing [1] to their attention, for pointing out that their results might be valid for generalized Fibonacci sequences and for generalized Lucas sequences, for carefully reading preliminary versions of this paper, and for pointing out Remark 2 as well as giving other helpful suggestions.

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AMS numbers: 11B39, 11A99

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