THE CONNECTIVITY OF A PARTICULAR GRAPH

Marc S. Ordower*
University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
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Let G be a graph with vertex set $N = \{1, 2, 3, \ldots\}$ and edge set E where $\{a, b\} \in E$ if and only if $a^2 + b^2 = c^2$ for some c in N. From the standard parameterization of Pythagorean triples, it is easy to deduce that 1 and 2 are isolated vertices and that 3 and 4 together comprise a connected component of G. Our result concerns the connectivity of the rest of the graph.

Theorem: $N \setminus \{1, 2, 3, 4\}$ is connected in the graph G.

Proof: One may verify that 8, 15, 20, 21, 72, 30, 16 is a path in G between 8 and 16. Note also that $\{a, b\} \in E$ implies that $\{ca, cb\} \in E$ for all $c \in N$. Therefore, by multiplying the elements in the above path by the appropriate power of 2, we find a path in G between 2^k and 2^{k+1} for all $k \geq 3$.

Next, given $n \geq 5$, we recursively find a path $P_n : n = n_0, n_1, \ldots, n_r = 2^k$ for some $k \geq 3$ according to the following algorithm: factor $n_i = p_i m_i$ where p_i is the largest prime factor of n_i; if $p_i = 2$ then we are done; otherwise, set $n_{i+1} = \frac{p_i^2 - 1}{2} \cdot m_i$.

We make two observations to verify that this algorithm generates the desired path. First, note that

$$n_i^2 + n_{i+1}^2 = (p_i m_i)^2 + \left(\frac{p_i^2 - 1}{2} \cdot m_i\right)^2 = \left(\frac{p_i^2 + 1}{2} \cdot m_i\right)^2$$

implies that $\{n_i, n_{i+1}\} \in E$.

Second, note that all prime factors of $\frac{p^2-1}{2}$ (p an odd prime) are strictly less than p. Hence, for all $i \in \{1, 2, \ldots, r-1\}$, if $n_i = p_i^{s_i} m_i'$ where $\gcd(p_i, m_i') = 1$, then $p_i = p_{i+1} = \cdots p_{i+s_i-1} > p_{i+s_i}$. Therefore, the algorithm terminates after a finite number of steps. □

Corollary: If H is the graph with vertex set N and edge set E' where $\{a, b\} \in E', a > b$ if and only if $a^2 - b^2 = c^2$ for some $c \in N$, then $N \setminus \{1, 2\}$ is connected.

Proof: One notes that, for all $\{a, b\} \in E$, there exists a $c \in N$ such that $\{a, c\}, \{b, c\} \in E'$. Also note that $\{3, 5\}, \{4, 5\} \in E'$. □

REFERENCE

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