MANDELBROT'S FUNCTIONAL ITERATION AND CONTINUED FRACTIONS

Michael I. Ratliff
Northern Arizona University, PO Box 5717, Flagstaff, AZ 86011-5717
(Submitted September 1991)

Functional iteration which gives rise to the Mandelbrot set is concerned with functions of the form $g(x) = x^2 + c$, where c is a point in the complex plane. This paper provides an algorithm which uses Newton's method and Mandelbrot-type functional iteration to produce a sequence of rational numbers that converges quadratically to the square-root of any given positive integer, and has a best approximation property. The algorithm is then modified so that convergence can be accelerated to any power of 2 order.

NEWTON'S METHOD VERSUS CONTINUED FRACTIONS

Let n be any given positive integer that is not a perfect square. We can find $\sqrt{n}$ by using Newton's method with the equation $f(x) = x^2 - n = 0$. For an arbitrary choice of positive x_0, the sequence

$$x_{k+1} = x_k - \frac{x_k^2 - n}{2x_k}, \text{ or } x_{k+1} = \frac{x_k^2 + n}{2x_k}, \; k \geq 0, \tag{1}$$

will converge quadratically to the square-root of n. We say the sequence $\{t_n\}$ converges linearly or quadratically ($\alpha = 1$ or 2) to t if:

(i) $\lim\limits_{n\to\infty} t_n = t$,

(ii) $\lim\limits_{n\to\infty} \frac{|t_{n+1} - t|}{|t_n - t|^\alpha} = \lambda$, where $0 < \lambda < \infty$. [4]

On the other hand, using continued fractions, we can obtain a sequence of rational numbers, p_k / q_k, which converges linearly to the square-root of n. Each of these methods of obtaining $\sqrt{n}$ has its advantages. Newton's method converges faster than does the continued fraction method, but continued fractions have the best possible approximation property. That is,

$$\left|\sqrt{n} - \frac{p}{q}\right| < \left|\sqrt{n} - \frac{p_k}{q_k}\right| \text{ implies that } q > q_k.$$

What we seek to do is find a method that has the advantages of both Newton's method and the continued fractions method.

NEWTON'S METHOD AND MANDELBROT ITERATION

Let $y_k = x_k^2 - n$, then (1) implies

$$y_{k+1} = x_{k+1}^2 - n = \left(\frac{x_k^2 + n}{2x_k}\right)^2 - n = \left(\frac{x_k^2 - n}{2x_k}\right)^2 = \frac{y_k^2}{4(y_k + n)}.$$

Let $y_k = 1/z_k$ and invert the above equation to obtain $z_{k+1} = 4z_k + 4nz_k^2$, which becomes

$$\frac{w_{k+1}}{4n} = \frac{4w_k}{4n} + 4n\frac{w_k^2}{16n^2}, \text{ if we let } z_k = \frac{w_k}{4n}.$$

Finally, setting $w_k = v_k - 2$, the equation above becomes

$$v_{k+1} = v_k^2 - 2. \tag{2}$$

But this is just Mandelbrot-type iteration with functions of the form $g(x) = x^2 + c$, where in this case $c = -2$.

RESULTS FROM THE THEORY OF CONTINUED FRACTIONS

The following points are known from the theory of continued fractions [2]:

(i) The continued fraction expansion of $\sqrt{n} = \langle a_0, \overline{a_1, a_2, \ldots, a_{r-1}, 2a_0}\rangle$. (3)

(ii) There exists a smallest subscript, s, such that the convergent p_s/q_s has the property

$$p_s^2 - nq_s^2 = 1. \tag{4}$$

(iii) $s = (1 + r \bmod 2)r - 1$.

(iv) If p, q are any integers that satisfy the equation $p^2 - nq^2 = 1$, then p/q is actually a continued fraction convergent to $\sqrt{n}$, say $p = p_j$ and $q = q_j$.

(v) If t is a positive integer such that $j = s + (1 + r \bmod 2)(t-1)r$, then

$$p_j + \sqrt{n}\, q_j = (p_s + \sqrt{n}\, q_s)^t, \text{ and conversely.} \tag{5}$$

(vi) For all positive integers m, if $j = s(1 + r \bmod 2)mr$, then the continued convergent p_j/q_j satisfies (4).

A NEW SEQUENCE

Newton's method, (1), allows us arbitrary choice of x_0. This gives rise to an arbitrary choice of v_0 in (2). We seek to choose v_0 so that the sequence (2) is closely related to the sequence, p_k, given by the continued fraction expansion of $\sqrt{n}$.

Let us define the sequence $\{v_k\}$, $k = 1, 2, \ldots$ by letting

$$v_0 = 2p_s \text{ and } v_{k+1} = v_k^2 - 2, \text{ for } k \geq 0. \tag{6}$$

Notice that this definition is the same as (2). This leads us to the following theorem.

Theorem 1: The sequence $\{v_k\}$, $k \geq 0$, is the same as the sequence $\{2p_{j_k}\}$, where

$$j_k = s + (1 + r \bmod 2)(2^k - 1)r. \tag{7}$$

Before we prove Theorem 1, we state the following

Lemma: If $j_k = s + (1 + r \bmod 2)(2^k - 1)r$, $k \geq 0$, then

$$p_{j_{k+1}} = p_{j_k}^2 + nq_{j_k}^2. \tag{8}$$

Proof: From (v),

$$p_{j_{k+1}}+\sqrt{n}\,q_{j_{k+1}}=\left(p_s+\sqrt{n}\,q_s\right)^{2^{k+1}} \quad \text{and} \quad p_{j_k}+\sqrt{n}\,q_{j_k}=\left(p_s+\sqrt{n}\,q_s\right)^{2^k}$$

So,

$$\begin{aligned} p_{j_{k+1}}+\sqrt{n}\,q_{j_{k+1}} &= \left(p_{j_k}+\sqrt{n}\,q_{j_k}\right)^2 \\ &= p_{j_k}^2+nq_{j_k}^2+2p_{j_k}q_{j_k}\sqrt{n}. \end{aligned}$$

We obtain $p_{j_{k+1}}=p_{j_k}^2+nq_{j_k}^2$, by equating the rational parts of both sides of the previous equation.

We return to the proof of Theorem 1. We use induction to prove our result. When $k=0$, $j_0=s$, so $2p_{j0}=2p_s=v_0$, by definition 6. Let us assume the result holds for some positive integer k. Then

$$v_{k+1}=v_k^2-2=(2p_{j_k})^2-2=2(2p_{j_k}^2-1). \tag{9}$$

From (vi), we conclude that

$$2p_{j_k}^2-1=p_{j_k}^2+nq_{j_k}^2=p_{j_{k+1}}, \text{ using (8).}$$

Putting this into (9), it follows that $v_{k+1}=2p_{j_{k+1}}$, and the theorem is proved.

QUADRATIC CONVERGENCE

It is also known from the theory of continued fractions [1] that, if p/q is a convergent for $\sqrt{n}$, then

$$\frac{c(n)}{q^2}<\left|\sqrt{n}-\frac{p}{q}\right|\le\frac{1}{q^2}, \text{ where } c(n)>0.$$

That is, the error in estimation of $\sqrt{n}$ by p/q is of the order $1/q^2$. Now, for the sequence of approximations p_{j_k}/q_{j_k} (which are also continued fraction convergents), we have

$$\frac{(\text{error at stage } (k+1))}{(\text{error at stage } k)^2}\cong\frac{\dfrac{1}{q_{j_{k+1}}^2}}{\left(\dfrac{1}{q_{j_k}^2}\right)^2}\cong\frac{1}{4n}\,\frac{(v_{k+1}-2)^2}{v_{k+1}^2-4}, \text{ using (2), (4), and Theorem 1.} \tag{10}$$

Since $v_k\to\infty$ as $k\to\infty$, the right-hand side of the last equation converges to $1/4n$; this confirms that p_{j_k}/q_{j_k} converges to $\sqrt{n}$ quadratically.

THE ALGORITHM

1. Let n be given, n a positive integer that is not a perfect square.
2. Use the continued fractions algorithm for $\sqrt{n}$ [3] to find
   (i) $\sqrt{n}=\langle a_0,\overline{a_1,\ldots,a_{r-1},2a_0}\rangle$, and
   (ii) p_s, q_s,, where s is the smallest positive integer such that $p_s^2-nq_s^2=1$ $(s=(1+r \bmod 2)r-1)$.

3. $v_0 = 2p_s$, and $v_{k+1} = v_k^2 - 2$, for $k \geq 0$.

4. Define $j_k = s + (1 + r \bmod 2)(2^k - 1)r$, then

$$p_{j_k} = \frac{v_k}{2}, \text{ and } q_{j_k} = \sqrt{\frac{p_{j_k}^2 - 1}{n}}$$

are such that p_{j_k}/q_{j_k} is a continued fraction convergent to $\sqrt{n}$, and the sequence p_{j_k}/q_{j_k} converges quadratically to $\sqrt{n}$.

AN EXAMPLE

Let us consider an example of the algorithm where $n = 19$. Then we can use the continued fraction algorithm to find:

$$\sqrt{19} = \langle 4, \overline{2, 1, 3, 1, 2, 8} \rangle.$$

In this case, we notice that $r = 6$, and that $s = (1 + r \bmod 2)r - 1 = 5$. Thus, again using the continued fraction algorithm for convergents, we obtain:

j			0	1	2	3	4	5	6	$\cdots$
a			4	2	1	3	1	2	8	
p	0	1	4	9	13	48	61	170	1421	$\cdots$
q	1	0	1	2	3	11	14	39	326	

Since s=5, the convergent $p_5/q_5 = 170/39$ satisfies (4). Thus, $v_0 = 340$, and $v_1 = g(v_0) = 340^2 - 2 = 11598$, which implies that

$$p_{j_1} = 57799 \quad \text{and} \quad q_{j_1} = \sqrt{\frac{p_{j_1}^2 - 1}{19}} = 13260.$$

Similarly, $v_2 = g(115598) = 115598^2 - 2 = 13362897602$, and

$$p_{j_2} = \frac{v_2}{2} = 6681448801 \quad \text{and} \quad q_{j_2} = \sqrt{\frac{p_{j_1}^2 - 1}{19}} = 1532829480.$$

Continuing in this manner, we obtain sequences $\{p_{j_k}\}$ and $\{q_{j_k}\}$ of integers such that the sequence $\{p_{j_k}/q_{j_k}\}$ is a sequence of continued fraction convergents which converge to $\sqrt{19}$ with quadratic order.

CONCLUSION

We note that we obtained the sequence ,$v_{k+1} = v_k^2 - 2, k \geq 0$, by iteration of the function $g(x) = x^2 - 2$ and starting at $v_0 = 2p_s$. We can ask, "What happens if we iterate g twice?" That is, let us define the sequence

$$u_{k+1} = g^2(u_k) = g(g(u_k)) = u_k^4 - 4u_k^2 + 2, k \geq 0,$$

with $u_0 = 2p_s$. Using arguments similar to the ones given above, we can show that $u_k = 2p_{\ell_k}$, where

$$\ell_k = s + (1 + r \bmod 2)(2^{2k} - 1)r.$$

Furthermore, if q_{ℓ_k} is obtained from p_{ℓ_k} by the use of (4), then p_{ℓ_k}/q_{ℓ_k} define a sequence of continued fraction convergents which converge to n with order 2^2.

These methods generalize to prove

Theorem 2: Let m be a positive integer, and let us define the sequence $\{t_k\}$, $k \geq 0$, as follows:

$$t_0 = 2p_s, \quad t_{k+1} = g^m(t_k),$$

where $g(x) = x^2 - 2$ and g^m is the m-fold iteration of g. Then $t_k = 2p_{h_k}$, where

$$h_k = s + (1 + r \bmod 2)(2^{mk} - 1)r.$$

Also, if

$$q_{h_k} = \sqrt{\frac{p_{h_k}^2 - 1}{n}},$$

then the sequence

$$\frac{p_{h_k}}{q_{h_k}}, k \geq 0,$$

is a sequence of continued fraction convergents which converge to $\sqrt{n}$ with order of convergence equal to 2^m.

REFERENCES

1. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*, 5th ed., pp. 138 and 161. Oxford: Oxford Press, 1979.
2. I. Niven & H. Zuckerman. *An Introduction to the Theory of Numbers*, 4th ed., pp. 204-12. New York: Wiley, 1979.
3. Ibid., pp. 215-16.
4. R. L. Burden & J. D. Faires. *Numerical Analysis*, 4th ed., p. 60. London: PWS-Kent, 1989.

AMS numbers: 11—Number Theory, 65—Numerical Analysis, 68—Computer Science

❖❖❖

EDITOR ON LEAVE OF ABSENCE

The Editor has been asked to visit Yunnan Normal University in Kunming, China, for the Fall semester of 1993. This is an opportunity that the Editor and his wife feel cannot be turned down. They will be in China from August 1, 1993, until approximately January 10, 1994. The August and November issues of *The Fibonacci Quarterly* will be delivered to the printer early enough so that these two issues can be published while the Editor is out of the country. The Editor has also arranged for several individuals to send out articles to be refereed which have been submitted for publication in *The Fibonacci Quarterly* or submitted for presentation at the ***Sixth International Conference on Fibonacci Numbers and Their Applications.*** Things may be a little slower than normal, but every attempt will be made to insure that all goes as smoothly as possible while the Editor is on leave in China. **PLEASE CONTINUE TO USE THE NORMAL ADDRESS FOR SUBMISSION OF PAPERS AND ALL OTHER CORRESPONDENCE.**