SUMS OF POWERS OF DIGITAL SUMS

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1. INTRODUCTION

In a recent article [2], the authors showed that, for any positive integer k,

$$\frac{1}{x}\sum_{n\le x} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O\left(\log^{k-\frac{1}{3}} x\right),$$

where $s(n)$ denotes the digital sum of the nonnegative integer n and $\log x$ denotes the base 10 logarithm of x. It was conjectured that, for any positive integer k,

$$\frac{1}{x}\sum_{n\le x} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O\left(\log^{k-1} x\right).$$

During a presentation of this result, Carl Pomerance asked if there was any evidence for this better "big-oh" term. At the time, the conjecture was based solely on two results, one by Cheo and Yien [1] which states that

$$\frac{1}{x}\sum_{n\le x} s(n) = \frac{9}{2}\log x + O(1)$$

and the other by Kennedy and Cooper [3] which states that

$$\frac{1}{x}\sum_{n\le x} s(n)^2 = \left(\frac{9}{2}\right)^2 \log^2 x + O(\log x).$$

In this article we will show that

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^k = \left(\frac{9}{2}\right)^k n^k + O\left(n^{k-1}\right).$$

This provides more evidence for this better "big-oh" term.

In [1], Cheo and Yien found that

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i) = \frac{9}{2}n.$$

In a similar manner, Kennedy and Cooper [3] showed that

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^2 = \frac{81}{4}n^2 + \frac{33}{4}n.$$

Furthermore, using MAPLE, the following formulas were calculated:

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^3 = \frac{729}{8}n^3 + \frac{891}{8}n^2,$$

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^4 = \frac{6561}{16}n^4 + \frac{8019}{8}n^3 + \frac{3267}{16}n^2 - \frac{3333}{40}n,$$

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^5 = \frac{59049}{32}n^5 + \frac{120285}{16}n^4 + \frac{147015}{32}n^3 - \frac{29997}{16}n^2,$$

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^6 = \frac{531441}{64}n^6 + \frac{3247695}{64}n^5 + \frac{3969405}{64}n^4 - \frac{1080783}{64}n^3 - \frac{329967}{32}n^2 + \frac{15873}{4}n,$$

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^7 = \frac{4782969}{128}n^7 + \frac{40920957}{128}n^6 + \frac{83357505}{128}n^5 - \frac{56133}{128}n^4 - \frac{20787921}{64}n^3 + \frac{999999}{8}n^2,$$

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^8 = \frac{43046721}{256}n^8 + \frac{122762871}{64}n^7 + \frac{750217545}{128}n^6 + \frac{76284747}{32}n^5 - \frac{1372208607}{256}n^4$$
$$+\frac{67777479}{64}n^3 + \frac{371095263}{320}n^2 - \frac{33333333}{80}n.$$

These results were obtained by initially considering the function

$$f(x) = (1+x+x^2+\cdots+x^9)^n.$$

We then repeatedly differentiated, multiplied by x, and substituted $x = 1$. However, when an exponent of 9 was used, the computation became too big for the memory of the computer. Nevertheless, these results reinforced our belief that the conjecture is true. We proceeded to delve more deeply into the generating function.

2. HIGHER DERIVATIVES

Because of the form of the function which was initially differentiated, i.e.,

$$(1+x+x^2+\cdots+x^9)^n,$$

we set out to find a formula for

$$\frac{d^m}{dx^m} g(x)^n,$$

where n and m are positive integers and g is an arbitrary, continuously differentiable function. After investigating the situation using the computer algebra system DERIVE, we noticed the following pattern.

Lemma 1: Let n and m be positive integers and g be a continuously differentiable function. Then

$$\frac{d^m}{dx^m} g^n = \sum_{n_1+2n_2+\cdots+mn_m=m} n(n-1)\cdots(n-n_1-\cdots-n_m+1)g^{n-n_1-\cdots-n_m}$$
$$\cdot\frac{m!}{(1!)^{n_1}n_1!(2!)^{n_2}n_2!\cdots(m!)^{n_m}n_m!}(g^{(1)})^{n_1}(g^{(2)})^{n_2}\cdots(g^{(m)})^{n_m},$$

where $n_1, n_2, \ldots, n_m$ are nonnegative integers.

The proof of this result is by induction on m. However, it might be noted here that Lemma 1 is just a special case of Faá di Bruno's formula [4] which states that if $f(x)$ and $g(x)$ are functions for which all the necessary derivatives are defined and m is a positive integer, then

$$\frac{d^m}{dx^m} f(g(x)) = \sum_{n_1+2n_2+\cdots+mn_m=m} \frac{m!}{n_1!\cdots n_m!}\left(\frac{d^{n_1+\cdots+n_m}}{dx^{n_1+\cdots+n_m}} f\right)(g(x))$$

$$\cdot\left(\frac{\frac{d}{dx}g(x)}{1!}\right)^{n_1}\cdots\left(\frac{\frac{d^m}{dx^m}g(x)}{m!}\right)^{n_m},$$

where $n_1, n_2, \ldots, n_m$ are nonnegative integers.

3. MAIN RESULT

We will need one final lemma before we can state and prove the main result. To do this, we let

$$f_0(x) = (1+x+x^2+\cdots+x^9)^n$$

and for any positive integer k,

$$f_k(x) = x \cdot f'_{k-1}(x).$$

Using f_k, we have the identity

$$\sum_{i=0}^{10^n-1} s(i)^k = f_k(1).$$

With these definitions in mind, we can state the following lemma.

Lemma 2: For any positive integer m,

$$f_m(x) = \sum_{i=1}^{m} \left\{ {m \atop i} \right\} x^i f_0^{(i)}(x),$$

where $\{\cdot\}$ denotes a Stirling number of the second kind.

Proof: We shall prove this result by induction on m. The result is clearly true for $m = 1$. Now assume that the result is true for any positive integer $m \geq 1$. By the definition of f_{m+1} and the induction hypothesis, we have

$$f_{m+1}(x) = x \cdot f'_m(x) = x \cdot \frac{d}{dx}\left(\sum_{i=1}^{m} \left\{ {m \atop i} \right\} x^i f_0^{(i)}(x)\right).$$

Next, by the product rule and simplification, we have

$$x \cdot \frac{d}{dx}\left(\sum_{i=1}^{m} \left\{ {m \atop i} \right\} x^i f_0^{(i)}(x)\right) = x \cdot \sum_{i=1}^{m} \left\{ {m \atop i} \right\} \left(x^i f_0^{(i+1)}(x) + f_0^{(i)}(x) \cdot ix^{i-1}\right)$$

$$= \sum_{i=1}^{m} \left(\left\{ {m \atop i} \right\} x^{i+1} f_0^{(i+1)}(x) + \left\{ {m \atop i} \right\} \cdot ix^i f_0^{(i)}(x)\right).$$

Finally, by simplification and the fact that

$$\left\{ {m \atop i-1} \right\} + i\left\{ {m \atop i} \right\} = \left\{ {m+1 \atop i} \right\},$$

we have that

$$\begin{aligned}
&\sum_{i=1}^{m}\left(\left\{ {m \atop i} \right\} x^{i+1} f_0^{(i+1)}(x) + \left\{ {m \atop i} \right\} \cdot i x^i f_0^{(i)}(x)\right) \\
&= \left\{ {m \atop 1} \right\} x f_0^{(1)}(x) + \sum_{i=2}^{m}\left(\left\{ {m \atop i-1} \right\} + i\left\{ {m \atop i} \right\}\right) x^i f_0^{(i)}(x) + \left\{ {m \atop m} \right\} x^{m+1} f_0^{(m+1)}(x) \\
&= \sum_{i=1}^{m+1}\left\{ {m+1 \atop i} \right\} x^i f_0^{(i)}(x).
\end{aligned}$$

Thus, the result is true for $m + 1$. Therefore, by induction, Lemma 2 is true for any positive integer m.

Finally, we have the main theorem.

Theorem: For all positive integers n and k,

$$\sum_{i=0}^{10^n-1} s(i)^k = \left(\frac{9}{2}\right)^k n^k 10^n + O(n^{k-1}10^n).$$

Proof: We first use Lemma 2 to obtain

$$\sum_{i=0}^{10^n-1} s(i)^k = f_k(1) = \sum_{i=1}^{k}\left\{ {k \atop i} \right\} 1^i f_0^{(i)}(1).$$

Next, by Lemma with $g^n = f_0$ and the fact that $f_0(x) = (1 + x + x^2 + \cdots + x^9)^n$, we have that

$$\sum_{i=1}^{k}\left\{ {k \atop i} \right\} 1^i f_0^{(i)}(1) = n^k 10^{n-k} 45^k + O(n^{k-1}10^n) = \left(\frac{9}{2}\right)^k n^k 10^n + O(n^{k-1}10^n).$$

This proves our main result.

4. QUESTIONS

We conclude this paper with some open questions:

Can we find an exact formula for

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^9$$

and is there a general exact formula for

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^k$$

for all positive integers n and k? Finally, despite the fact that we now have more compelling evidence, we still have not established the conjecture that, for any positive integer k,

$$\frac{1}{x}\sum_{n\le x} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O(\log^{k-1} x).$$

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