ON THE STRUCTURE OF THE SET OF DIFFERENCE SYSTEMS DEFINING (3, F) GENERALIZED FIBONACCI SEQUENCES

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The $(2, F)$ generalized Fibonacci sequences were defined in [1] and [2]. In [3] K. Atanassov extended the definition to the case of three sequences and listed thirty-six systems defining the $(3, F)$ generalized Fibonacci sequences. Ten of these thirty-six systems were discarded as trivial and the remaining twenty-six were placed in seven classes termed "groups." In this paper the structure of the systems of three second-order difference equations defining the $(3, F)$ generalized Fibonacci sequences is developed. This development is based on the following definitions of the permutations on the letters a, b, and c:

$$i\colon \begin{array}{l} a\to a \\ b\to b \\ c\to c \end{array} \qquad \alpha\colon \begin{array}{l} a\to b \\ b\to a = (ab) \\ c\to c \end{array} \qquad \beta\colon \begin{array}{l} a\to c \\ b\to b = (ac) \\ c\to a \end{array}$$

$$\gamma\colon \begin{array}{l} a\to a \\ b\to c = (bc) \\ c\to b \end{array} \qquad \delta\colon \begin{array}{l} a\to c \\ b\to a = (acb) \\ c\to b \end{array} \qquad \varepsilon\colon \begin{array}{l} a\to b \\ b\to c = (abc) \\ c\to a \end{array}$$

Note that $\delta = \alpha\beta$ where $\alpha\beta$ indicates that the permutation β is applied first, followed by α. Similarly, $\varepsilon = \beta\alpha$.

These definitions give rise to the following multiplication table for the six permutations:

$\cdot$	i	α	β	γ	δ	ε
i	i	α	β	γ	δ	ε
α	α	i	δ	ε	β	γ
β	β	ε	i	δ	γ	α
γ	γ	δ	ε	i	α	β
δ	δ	γ	α	β	ε	i
ε	ε	β	γ	α	i	δ

The six permutations of the letters a, b, and c form a group which is isomorphic to the symmetric group S_3. The group of permutations of the letters a, b, and c will be denoted by S_c.

Using these preliminaries, the $(3, F)$ generalizations of the Fibonacci sequence may be defined.

Definition: Let $C_i, 1 \le i \le 6$, be six real numbers; $X_0 = \{a_0, b_0, c_0\} = \{C_1, C_2, C_3\}$; $X_1 = \{a_1, b_1, c_1\} = \{C_4, C_5, C_6\}$; and let ρ, σ, and τ be permutations of S_c. Then the solutions

$$X = \langle X_i \rangle_0^\infty = \langle \{a_i, b_i, c_i\} \rangle_0^\infty = \left\{ \langle a_i \rangle_0^\infty, \langle b_i \rangle_0^\infty \langle c_i \rangle_0^\infty \right\}$$

of the difference system

$$\rho X_{i+2} = \sigma X_{i+1} + \tau X_i, \quad i \geq 0, \tag{1}$$

with initial conditions X_0, X_1, are the $(3, F)$ generalizations of the Fibonacci sequence. Since there are six permutations in S_c, there are a total of 216 systems of form (1). The systems of form (1) can be represented by ordered triples of permutations of S_c. Thus,

$$(\rho, \sigma, \tau) \text{ represents } \rho X_{i+2} = \sigma X_{i+1} + \tau X_i, \quad i \geq 0.$$

Consequently, the triple (i, δ, ε) represents the equations

$$\begin{aligned} a_{i+2} &= c_{i+1} + b_i \\ b_{i+2} &= a_{i+1} + c_i, \quad i \geq 0, \\ c_{i+2} &= b_{i+1} + a_i \end{aligned}$$

which is S_{30} in Atanassov [3]. Two different systems (ρ, σ, τ) and (ρ', σ', τ') may not define distinct $(3, F)$ sequences. For example, with given initial conditions X_0, X_1, the system

$$(\varepsilon, i, \delta) = \begin{cases} b_{i+2} = a_{i+1} + c_i \\ c_{i+2} = b_{i+1} + a_i, \quad i \geq 0, \\ a_{i+2} = c_{i+1} + b_i \end{cases}$$

defines the same sequence as $S_{30} = (i, \delta, \varepsilon)$ since the same equations determine the successive terms of the sequences. Observe that the two systems (i, δ, ε) and (ε, i, δ) are row equivalent. In general, two systems (ρ, σ, τ) and (ρ', σ', τ') are row equivalent if and only if one system can be obtained from the other by multiplication of the permutations of the other system by the same permutation. That is,

Definition: Let $\rho, \sigma, \tau, \rho', \sigma', \tau'$ be six permutations of S_c. Then the systems (ρ, σ, τ) and (ρ', σ', τ') are row equivalent if there exists a permutation η in S_c such that $\eta(\rho, \sigma, \tau) = (\eta\rho, \eta\sigma, \eta\tau) = (\rho', \sigma', \tau')$.

Since there are six permutations in S_c, there are six systems that are row equivalent to a given system (ρ, σ, τ). Thus, the 216 systems are partitioned into thirty-six equivalence classes of row equivalent systems which are the systems considered by Atanassov in [3]. For example, the systems S_{30} and S_{22} of Atanassov are

$$[S_{30}] = [(i, \delta, \varepsilon)] = \{(i, \delta, \varepsilon), (\varepsilon, i, \delta), (\alpha, \beta, \gamma), (\gamma, \alpha, \beta), (\beta, \gamma, \alpha), (\delta, \varepsilon, i)\}, \text{ and}$$

$$[S_{22}] = [(i, \alpha, \beta)] = \{(i, \alpha, \beta), (\alpha, i, \delta), (\beta, \varepsilon, i), (\gamma, \delta, \varepsilon), (\delta, \gamma, \alpha), (\varepsilon, \beta, \gamma)\},$$

where $[(\rho, \sigma, \tau)]$ indicates the equivalence class of (ρ, σ, τ). Since each equivalence class contains one system that has the identity as the first permutation, the classes may be uniquely represented by an ordered pair of permutations (ϕ, ψ) where ϕ and ψ are permutations of S_c.

A relation is now defined on the equivalence classes of row equivalent systems.

Definition: Let ϕ, ψ, ϕ', and ψ' be permutations of S_c. The ordered pair (ϕ, ψ) is equivalent to the ordered pair (ϕ', ψ'), written $(\phi, \psi) \equiv (\phi', \psi')$, if there exists a η in S_c such that $\phi' = \eta\phi\eta^{-1}$ and $\psi' = \eta\psi\eta^{-1}$.

Since $\phi = i\phi i^{-1}$ and $\psi = i\psi i^{-1}$, the relation is reflexive. Suppose $(\phi, \psi) \equiv (\phi', \psi')$. Then, for some μ and μ^{-1} in S_c, $\phi' = \mu\phi\mu^{-1}$ and $\psi' = \mu\psi\mu^{-1}$. Therefore, for $\eta = \mu^{-1}, \phi = \eta\phi'\eta^{-1}$ and $\psi = \eta\psi'\eta^{-1}$. Hence, the relation is symmetric. Suppose $(\phi, \psi) \equiv (\phi', \psi')$ and $(\phi', \psi') \equiv (\phi'', \psi'')$. Then, for some η and μ in S_c, $\phi' = \eta\phi\eta^{-1}, \psi' = \eta\psi\eta^{-1}, \phi'' = \mu\phi'\mu^{-1}$, and $\psi'' = \mu\psi'\mu^{-1}$. Consequently, $\phi'' = \mu\phi'\mu^{-1} = \mu\eta\phi\eta^{-1}\mu^{-1} = \rho\phi\rho^{-1}$ and $\psi'' = \mu\psi'\mu^{-1} = \mu\eta\psi'\eta^{-1}\mu^{-1} = \rho\psi\rho^{-1}$ for $\rho = \mu\eta$. Hence, by definition, $(\phi, \psi) \equiv (\phi'', \psi'')$, and the relation is transitive. Thus, the relation is an equivalence relation. The definition of equivalent systems requires that there exists η in S_c such that ϕ and ϕ', ψ and ψ' belong to the same conjugate classes for that η.

It is well known that the conjugate classes of S_3 are the permutations with the same cycle structure (see [6]). Since S_c is isomorphic to S_3, the conjugate classes of S_c are: $C\{i\} = \{i\}$, $C\{\alpha\} = \{\alpha, \beta, \gamma\}$, $C\{\delta\} = \{\delta, \varepsilon\}$, where $C\{\sigma\}$ denotes the conjugate class of σ. Let $\overline{(\phi, \psi)}$ denote the equivalence class of (ϕ, ψ). If ϕ and ψ belong to different conjugate classes, then the recursion systems in the equivalence class $\overline{(\phi, \psi)}$ are the ordered pairs with ϕ a member of $C\{\phi\}$ and ψ a member of $C\{\psi\}$. Thus, there is one equivalence class for each pair of conjugate classes in S_c. The classes and the corresponding schemes of Atanassov [3] are:

$$\overline{(i, \alpha)} = \{(i, \alpha), (i, \beta), (i, \gamma)\} = \{S_5, S_{10}, S_2\},$$
$$\overline{(i, \delta)} = \{(i, \delta), (i, \varepsilon)\} = \{S_9, S_6\},$$
$$\overline{(\alpha, i)} = \{(\alpha, i), (\beta, i), (\gamma, i)\} = \{S_{13}, S_{27}, S_3\},$$
$$\overline{(\delta, i)} = \{(\delta, i), (\varepsilon, i)\} = \{S_{25}, S_{15}\},$$
$$\overline{(\alpha, \delta)} = \{(\alpha, \delta), (\beta, \delta), (\gamma, \delta), (\beta, \varepsilon), (\gamma, \varepsilon)\} = \{S_{21}, S_{35}, S_{11}, S_{18}, S_{32}, S_8\},$$
$$\overline{(\delta, \alpha)} = \{(\delta, \alpha), (\delta, \beta), (\delta, \gamma), (\varepsilon, \alpha), (\varepsilon, \beta), (\varepsilon, \gamma)\} = \{S_{29}, S_{34}, S_{26}, S_{19}, S_{24}, S_{16}\}.$$

If ϕ and ψ belong to the same conjugate class of S_c, and $\phi = \psi$, then ϕ' and ψ' must also belong to the same conjugate class and $\phi' = \psi'$. Consequently, there are as many classes of this type as there are conjugate classes in S_c, namely, three. Moreover, there are as many systems in each class as there are permutations in $C\{\phi\}$. The classes of this type are:

$$\overline{(i, i)} = \{(i, i)\} = \{S_1\},$$
$$\overline{(\alpha, \alpha)} = (\alpha, \alpha), (\beta, \beta), (\gamma, \gamma)\} = \{S_{17}, S_{36}, S_4\},$$
$$\overline{(\delta, \delta)} = \{(\delta, \delta), (\varepsilon, \varepsilon)\} = \{S_{33}, S_{20}\}.$$

If ϕ and ψ belong to the same conjugate class, but $\phi \neq \psi$, then ϕ' and ψ' are distinct and also belong to the same conjugate class. There are as many equivalence classes of this type as there are conjugate classes in S_c which contain at least two distinct permutations, namely, two. There are as many systems in each class as there are combinations of distinct permutations in one conjugate class. The systems of this type are:

$$\overline{(\alpha, \beta)} = \{(\alpha, \beta), (\alpha, \gamma), (\beta, \alpha), (\beta, \gamma), (\gamma, \alpha), (\gamma, \beta)\} = \{S_{22}, S_{14}, S_{31}, S_{28}, S_7, S_{12}\},$$
$$\overline{(\delta, \varepsilon)} = \{(\delta, \varepsilon), (\varepsilon, \delta)] = \{S_{30}, S_{23}\}.$$

Hence, the thirty-six systems defined by Atanassov [3] belong to eleven equivalence classes as listed above.

Theorem: Let (ϕ, ψ) and (ϕ', ψ') be two systems, let $\langle X_i \rangle_0^\infty$ and $\langle Z_i \rangle_0^\infty$ be solutions to (ϕ, ψ) and (ϕ', ψ'), respectively, and let $\eta X_0 = Z_0$ and $\eta X_1 = Z_1$. then (ϕ, ψ) and (ϕ', ψ') are equivalent systems if and only if $\langle \eta X_i \rangle_0^\infty = \langle Z_i \rangle_0^\infty$.

Proof: Suppose (ϕ, ψ) and (ϕ', ψ') are equivalent systems. Then $Z_2 = \phi' Z_1 + \psi' Z_0$. Since the systems are equivalent, for some η in S_c,

$$\begin{aligned} Z_2 &= \eta\phi\eta^{-1}Z_1 + \eta\psi\eta^{-1}Z_0 = \eta\phi X_1 + \eta\psi X_0 \\ &= \eta(\phi X_1 + \psi X_0) = \eta X_2. \end{aligned}$$

The theorem is true for $i = 2$. Assume that it is true for all $k \le n$ for some integer $n \ge 2$: $Z_{n+1} = \phi' Z_n + \psi' Z_{n-1}$. So again, since the systems are equivalent,

$$\begin{aligned} Z_{n+1} &= \eta\phi'\eta^{-1}Z_n + \eta\psi'\eta^{-1}Z_{n-1} \\ &= \eta\phi X_n + \eta\psi X_{n-1} = \eta X_{n+1}, \end{aligned}$$

the theorem holds for all $i \ge 0$.

Now assume that $\langle \eta X_i \rangle_0^\infty = \langle Z_i \rangle_0^\infty$. Then

$$X_{i+2} = \phi X_{i+1} + \psi X_i, \quad i \ge 0.$$

Since $\eta^{-1} Z_i = X_i$ for all a $i \ge 0$,

$$\eta^{-1}Z_{i+2} = \phi\eta^{-1}Z_{i+1} + \psi\eta^{-1}Z_i,$$

which is row equivalent to

$$\eta\eta^{-1}Z_{i+2} = \eta\phi\eta^{-1}Z_{i+1} + \eta\phi\eta^{-1}Z_i = Z_{i+2}, \quad i \ge 0.$$

But. $Z_{i+2} = \phi' Z_{i+1} + \psi' Z_i$ for $i \ge 0$. Therefore, $\phi' = \eta\phi\eta^{-1}$ and $\psi' = \eta\psi\eta^{-1}$, and the systems are equivalent. Thus, the theorem holds. As a result of the above theorem, only one system in each equivalence class need be solved since the solutions to the systems in an equivalence class are related to each other by a permutation of S_c. Therefore, all $(3, F)$ generalized Fibonacci sequences are determined by solving eleven systems. Furthermore, four of these systems, namely, $\overline{(i, i)}, \overline{(i, \alpha)}, \overline{(\alpha, i)}$, and $\overline{(\alpha, \alpha)}$, can be written in terms of generalized Fibonacci sequences and

$(2, F)$ generalized Fibonacci sequences. Generalized Fibonacci sequences are discussed in [4] and the $(2, F)$ generalized Fibonacci sequences are developed in [1], [2], [5], and [7]. Consequently, only seven new systems need to be solved in order to generate the solutions to all eleven equivalence classes.

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