FIBONACCI NUMBERS AND FRACTIONAL DOMINATION OF $P_m \times P_n$

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1. INTRODUCTION

The product of two paths, $P_m \times P_n$, is also known as the $m \times n$ complete grid graph, $G_{m,n}$, having vertex set $Z_m \times Z_n$, where Z_k denotes the set $\{1, 2, \ldots, k\}$. Two vertices, (i, j) and (r, s), are adjacent when $|i-r|+|j-s|=1$. Thus, $|V|=mn$ and $|E|=2mn-(m+n)$.

Let $G=(V, E)$ be a graph and $v \in V(G)$. Then the closed neighborhood of v, denoted $N[v]$, is the set $\{v\} \cup \{u \in V(G) | uv \in E(G)\}$.

The definition of fractional domination, as introduced by Hedetniemi et al. [3] is as follows: If g is a function mapping the vertex set, $V(G)$, into some set of real numbers, then for S a subset of $V(G)$, let $g(S)=\Sigma g(v)$ over all $v \in S$. Let $|g|=g(V(G))=g(v_1)+g(v_2)+\cdots+g(v_n)$. A real-valued function $g: V(G) \to [0,1]$ is a *fractional dominating function* if for every $v \in V(G)$, $g(N[v]) \geq 1$. A dominating function is *minimal* if for every $v \in V(G)$ with $g(v)>0$, there exists a vertex $u \in N[v]$ such that $g(N[u])=1$. The *fractional domination number* of G, denoted $\gamma_f(G)$, is the minimum, $|g|$, over all minimal dominating functions g.

A real-valued function $g: V(G) \to [0,1]$ is a *packing function* if, for every $v \in V(G)$ with $g(v)<1$, there exists a vertex $u \in N[v]$ where $g(N[u])=1$. Then the *(upper) fractional packing number* of G, denoted $P_f(G)$, is the maximum $|g|$ such that g is a maximal packing function.

The fractional parameters are related by the following.

Proposition 1.1: For every graph G, $P_f(G)=\gamma_f(G)$ (Domke [1]).

The formula of Proposition 1.2 computes the fractional domination number for $P_2 \times P_n$. No general formula is known for $\gamma_f(P_m \times P_n)$, for $m>2$, but fractional domination numbers for any graph may be computed using linear programming.

Proposition 1.2: $\gamma_f(P_2 \times P_n)=(n+1)/2+(\lceil n/2 \rceil - \lfloor n/2 \rfloor - 1)/(2n+2) = n/2+\lceil n/2 \rceil/(n+1)$.

Proof: It has been shown that $\gamma_f(P_2 \times P_n)=\lceil n/2 \rceil=(n+1)/2$ when $n \equiv 1 \pmod 2$, and that $\gamma_f(P_2 \times P_n)=(n^2+2n)/2(n+1)$ when $n \equiv 0 \pmod 2$ (Hare [4]).

Values of fractional domination numbers for $P_m \times P_n$ for several small (m, n) pairs may also be found in [4] and [5]. It would be interesting if a formula could be found for the arbitrary $m \times n$ complete grid graphs, as has been found for the 2-packing number [2]. In the remainder of this paper we develop upper and lower bounds for the fractional domination number of $P_m \times P_n$..

2. BOUNDS FOR THE FRACTIONAL DOMINATION NUMBER

Let $D_m=3F_1F_m+F_2F_{m-1}=3F_m+F_{m-1}$, where D stands for "denominator." We denote a vertex in the i^{th} row and j^{th} column of $G_{m,n}$ $(=P_m \times P_n)$ by $v_{i,j}$. The following develops upper and lower bounds for $\gamma_f(P_m \times P_n)$ which depend only on m.

Proposition 2.3: Let $m > 2, n > 2, 1 < j < n$, and $g(v_{i,j}) = F_i F_{m-i+1} / D_m$. Then

$$g(N[v_{1,j}]) = 1 = g(N[v_{2,j}]).$$

Proof:

$$\begin{aligned} g(N[v_{1,j}]) &= g(v_{1,j-1}) + g(v_{1,j}) + g(v_{1,j+1}) + g(v_{2,j}) \\ &= 3F_1F_m / D_m + F_2F_{m-1} / D_m = 1. \end{aligned}$$

$$\begin{aligned} g(N[v_{2,j}]) &= g(v_{1,j}) + g(v_{2,j-1}) + g(v_{2,j}) + g(v_{2,j+1}) + g(v_{3,j}) \\ &= (F_1F_m + 3F_2F_{m-1} + F_3F_{m-2}) / D_m. \end{aligned}$$

Since $3F_1F_m + F_2F_{m-1} = F_1F_m + 3F_2F_{m-1} + F_3F_{m-2}$, it follows that $g(N[v_{2,j}]) = 1$. By symmetry,

$$g(N[v_{m,j}]) = g(N[v_{m-1,j}]) = 1.$$

Proposition 2.4: Let $m > 3, n > 2, 1 < i < m-1, 1 < j < n$, and $g(v_{i,j}) = F_i F_{m-i+1} / D_m$. Then, $g(N[v_{1,j}]) = g(N[v_{i,j}]) = 1$.

Proof:

$$\begin{aligned} g(N[v_{i,j}]) &= g(v_{i-1,j}) + g(v_{i,j-1}) + g(v_{i,j}) + g(v_{i,j+1}) + g(v_{i+1,j}) \\ &= (F_{i-1}F_{m-i+2} + 3F_iF_{m-i+1} + F_{i+1}F_{m-i}) / D_m. \end{aligned}$$

$$\begin{aligned} g(N[v_{i+1,j}]) &= g(v_{i,j}) + g(v_{i+1,j-1}) + g(v_{i+1,j}) + g(v_{i+1,j+1}) + g(v_{i+2,j}) \\ &= (F_iF_{m-i+1} + 3F_{i+1}F_{m-1} + F_{i+2}F_{m-i-1}) / D_m. \end{aligned}$$

Since $F_{i-1}F_{m-i+2} + 3F_iF_{m-i+1} + F_{i+1}F_{m-i} = F_iF_{m-i+1} + 3F_{i+1}F_{m-i} + F_{i+2}F_{m-i-1}$, it follows that

$$g(N[v_{i,j}]) = g(N[v_{i+1,j}]).$$

From Proposition 2.3, $g(N[v_{2,j}]) = 1$, so $g(N[v_{i,j}]) = 1$ for all i, $1 \le i \le m$.

Theorem 2.5: Let $C_m = g(v_{1,j}) + g(v_{2,j}) + \cdots + g(v_{m,j})$ where $g(v_{i,j}) = F_iF_{m-i+1} / D_m$. Then, when $m \ge 3$, the sum of the function values over all vertices in column j is given by C_m / D_m where $C_m = \Sigma_{i=1,m}(F_iF_{m-i+1})D_m$ and $\gamma_f(P_m \times P_n) \le nC_m + c_\gamma$, where $c_\gamma \le 2[m/3]F_m / D_m$.

Proof: Since $g(N[v_{i,j}]) = 1$ for $2 \le j \le n-1$, all vertices in columns 2 through $n-1$ are dominated. In order to dominate column 1, let $g(v_{i,1})$ be modified as follows:

For $1 < i < m$, let $\sigma = \max\{g(v_{i-1,j}), g(v_{i,j}), g(v_{i+1,j})\}$.

Case 1. $m \equiv 0 \bmod 3$.
If $i \equiv 2 \bmod 3$, then $g(v_{i,1}) = F_iF_{m-i+1} / D_m + \sigma$.

Case 2. $m \equiv 1 \bmod 3$.
If $2i = (m+1)$, then $g(v_{i,1}) = 2F_iF_{m-i+1} / D_m$.
Else If $[(i \equiv 2 \bmod 3)$ and $(2i < m+1)]$ or $[(i \equiv 0 \bmod 3)$ and $(2i > m+1)]$, then $g(v_{i,1}) = F_iF_{m-i+1} / D_m + \sigma$.

Case 3. $m \equiv 2 \bmod 3$.

If $2i = m$, then $g(v_{i,1}) = 2F_iF_{m-i+1} / D_m$.

Else If $[(i \equiv 2 \bmod 3)$ and $(2i < m)]$ or $[(i \equiv 1 \bmod 3)$ and $(2i - 2 > m)]$, then $g(v_{i,1}) = F_iF_{m-i+1} / D_m + \sigma$.

Observe that this assignment produces $g(N[v_{i,1}]) \geq 1$ for all vertices in column 1. To show that g is minimal, observe that $g(N[v_{i,j}]) = 1$ for $1 \leq i \leq m$ and $2 \leq j \leq n-1$, except when $g(v_{1,j}) \neq F_iF_{m-i+1} / D_m$. Thus, only the case when $g(v_{i,1}) \neq F_iF_{m-i+1} / D_m$ must be examined. In the above procedure, each modification produces an assignment such that $g(N[v_{i-1,1}]) = 1$, $g(N[v_{i,1}]) = 1$, or $g(N[v_{i+1,1}]) = 1$. Thus, g is minimal.

To also dominate the vertices of column n, let c_γ be twice the functional value added to column 1 by the above modification. It is straightforward to show by induction on i, $1 < i < m-1$, that $F_m = F_{i+1}F_{m-i} + F_iF_{m-i-1}$. Thus, $F_m > F_{i+1}F_{m-i}$. Let $j = i+1$. Then $F_m > F_jF_{m-j+1}$, which yields $2[m/3](F_m / D_m) \geq c_\gamma$.

Such a minimal dominating function is given for $P_3 \times P_n$ by:

$$g(v_{i,j}) = g(v_{3,j}) = 2/7, \text{ for } 1 \leq j \leq n,$$
$$g(v_{2,j}) = 1/7, \text{ for } 1 < j < n, \text{ and}$$
$$g(v_{2,1}) = g(v_{2,n}) = 3/7.$$

Thus, $\gamma_f(P_3 \times P_n) \leq n(5/7) + 4/7$.

3. BOUNDS FOR THE FRACTIONAL PACKING NUMBER

From Proppositions 2.3 and 2.4 and the definition of fractional packing, it is clear that when $g(v_{i,j}) = F_iF_{m-i+1} / D_m$ for all i and j, then g is a maximal packing function and $|g| = nC_m$. However, the following improved bounds are easily obtained.

Proposition 3.6:

$$P_f(P_3 \times P_n) \geq nC_3 + 2/7, \qquad \text{for } n \geq 3,\ C_3 = 5/7.$$
$$P_f(P_4 \times P_n) \geq nC_4 + 4/11, \qquad \text{for } n \geq 4,\ C_4 = 10/11.$$
$$P_f(P_5 \times P_n) \geq nC_5 + 8/18, \qquad \text{for } n \geq 5,\ C_5 = 20/18.$$
$$P_f(P_6 \times P_n) \geq nC_6 + 18/29, \qquad \text{for } n \geq 6,\ C_6 = 38/29.$$

Proof: The following assignments of g produce maximal packing functions.

For $P_3 \times P_n$:

$$g(v_{1,1}) = g(v_{1,n}) = g(v_{3,1}) = g(v_{3,n}) = 3/7 = F_4 / D_3,$$
$$g(v_{2,2}) = g(v_{2,n-1}) = 0, \text{ and}$$
$$g(v_{i,j}) = F_iF_{m-i+1} / D_3, \text{ otherwise.}$$

Thus, $P_f(P_3 \times P_n) \geq n(5/7) + 2/7$.

For every vertex in rows 1 and 3, $g[N(v_{i,j})] = 1$, except for columns 1 and n. However, $g[N(v_{2,1})] = g[N(v_{2,n}) = 1$, so g is maximal.

For $P_4 \times P_n$:

$$g(v_{1,1}) = g(v_{1,n}) = g(v_{4,1}) = g(v_{4,n}) = 5/11 = F_5 / D_4,$$
$$g(v_{2,1}) = g(v_{2,n}) = g(v_{3,1}) = g(v_{3,n}) = 3/11 = F_4 / D_4,$$
$$g(v_{2,2}) = g(v_{3,2}) = g(v_{2,n-1}) = g(v_{3,n-1}) = 0, \text{ and}$$
$$g(v_{i,j}) = F_i F_{m-i+1} / D_4, \text{ otherwise.}$$

For every vertex in rows 1 and 4, $g[N(v_{i,j})] = 1$, so g is maximal.

For $P_5 \times P_n$:

$$g(v_{1,1}) = g(v_{1,n}) = g(v_{5,1}) = g(v_{5,n}) = 8/18 = F_6 / D_5,$$
$$g(v_{2,1}) = g(v_{2,n}) = g(v_{4,1}) = g(v_{4,n}) = 5/18 = F_5 / D_5,$$
$$g(v_{2,2}) = g(v_{2,n-1}) = g(v_{4,2}) = g(v_{4,n-1}) = 0, \text{ and}$$
$$g(v_{i,j}) = F_i F_{m-i+1} / D_5, \text{ otherwise.}$$

For every vertex in rows 1, 3, and 5 except vertices $v_{3,2}$ and $v_{3,n-1}$, $g[N(v_{i,j})] = 1$, so g is maximal.

For $P_6 \times P_n$:

$$g(v_{1,1}) = g(v_{1,n}) = g(v_{6,1}) = g(v_{6,n}) = 13/29 = F_7 / D_6,$$
$$g(v_{2,1}) = g(v_{2,n}) = g(v_{4,1}) = g(v_{4,n}) = 8/29 = F_6 / D_6,$$
$$g(v_{2,2}) = g(v_{2,n-1}) = g(v_{4,2}) = g(v_{4,n-1}) = 0,$$
$$g(v_{3,1}) = g(v_{3,n}) = 8/29,$$
$$g(v_{4,1}) = g(v_{4,n}) = 7/29, \text{ and}$$
$$g(v_{i,j}) = F_i F_{m-i+1} / D_6, \text{ otherwise.}$$

For every vertex in rows 1, 3, 4, and 6 except $v_{3,2}, v_{4,2}, v_{3,n-1}$, and $v_{4,n-1}$, $g[N(v_{i,j})] = 1$, so g is maximal.

Theorem 3.7: When $m > 6, n \geq m$, $P_f(P_m \times P_n) \geq nC_m + 4(F_{m-1} / D_m)$.

Proof: For $P_m \times P_n$:

$$g(v_{1,1}) = g(v_{1,n}) = g(v_{m,1}) = g(v_{m,n}) = F_{m+1} / D_m,$$
$$g(v_{2,1}) = g(v_{2,n}) = g(v_{m-1,1}) = g(v_{m-1,n}) = F_m / D_m,$$
$$g(v_{3,1}) = g(v_{3,n}) = g(v_{m-2,1}) = g(v_{m-2,n}) = F_m / D_m,$$
$$g(v_{2,2}) = g(v_{m-1,2}) = g(v_{2,n-1}) = g(v_{m-1,n-1}) = 0, \text{ and}$$
$$g(v_{i,j}) = F_i F_{m-i+1} / D_m, \text{ otherwise.}$$

In column 1, $g[N(v_{1,1})] = g[N(v_{2,1})] = g[N(v_{m,1})] = g[N(v_{m-1,1})] = 1$. For all vertices in column 2 except $v_{2,2}, v_{3,2}, v_{m-1,2}$, and $v_{m-2,2}$, $g[N(v_{i,2})] = 1$. For all vertices in colums 3 through $n-3$, $g[N(v_{i,j})] = 1$. Thus, every vertex is adjacent to some vertex (possibly itself) with $g[N(v_{i,j}] = 1$ and g is maximal. Column summations yield a net gain of $4F_{m-1} / D_m$.

Corollary 3.8: When $m > 6, n \geq m$, then $P_f(P_m \times P_n) \geq mn/5 + (2n/5)(F_m / D_m) + 4(F_{m-1} / D_m)$.

Proof: It is well known that, for $m \geq 4$,

$$C_m = \sum_{i=1,m} (F_i F_{m-i+1}) / D_m = ((3m+2)F_m + mF_{m-1})/5 = (m(3F_m + F_{m-1}) + 2F_m)/5.$$

Then

$$\begin{aligned} P_f(P_m \times P_n) &\geq nC_m + 4(F_{m-1}/D_m) \\ &= mn/5 + (2n/5)(F_m/D_m) + 4(F_{m-1}/D_m). \end{aligned}$$

The recurrence $C_m = F_m / D_m + C_{m-1} + C_{m-2}$ follows immediately and, for large m, C_m is approximately $m/5 + 0.145$.

4. CONCLUDING REMARKS

It has been shown in this paper that

$$\begin{aligned} n(5/7) + 2/7 \leq \gamma_f(P_3 \times P_n) \leq n(5/7) + 4/7, \\ n(10/11) + 4/11 \leq \gamma_f(P_4 \times P_n) \leq n(10/11) + 12/11, \\ n(20/18) + 8/18 \leq \gamma_f(P_5 \times P_n) \leq n(20/18) + 20/18, \\ n(38/29) + 18/29 \leq \gamma_f(P_6 \times P_n) \leq n(38/29) + 32/29 \end{aligned}$$

and, for $m > 6, n \geq m$,

$$nC_m + 4(F_{m-1}/D_m) \leq \gamma_f(P_m \times P_n) \leq nC_m + 2\lceil m/3 \rceil (F_m / D_m),$$

where $C_m = \Sigma_{i=1,m}(F_i F_{m-i+1})/D_m$ and $D_m = 3F_m + F_{m-1}$.

Although the methods of linear programming can be used to calculate γ_f for individual graphs, no exact construction is known for $\gamma_f(P_m \times P_n)$ for $m > 2$ Thus, the bounds presented in this paper provide a useful addition to our knowledge of domination parameters on grid graphs.

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